

Well-posedness and ill-posedness of the 3D generalized Navier-Stokes equations in Triebel-Lizorkin spaces

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Abstract

In this paper, we study the Cauchy problem of the 3-dimensional (3D) generalized incompressible Navier-Stokes equations (gNS) in Triebel-Lizorkin space $\dot{F}_{q_\alpha}^{-\alpha,r}(\mathbb{R}^3)$ with $(\alpha, r) \in (1, \frac{5}{4}) \times [2, \infty]$ and $q_\alpha = \frac{3}{\alpha-1}$. Our work establishes a *dichotomy* of well-posedness and ill-posedness depending on $r = 2$ or $r > 2$. Specifically, by combining the new endpoint bilinear estimates in $L_x^{q_\alpha} L_T^2$ with the characterization of Triebel-Lizorkin space via fractional semigroup, we prove the well-posedness of the gNS in $\dot{F}_{q_\alpha}^{-\alpha,r}(\mathbb{R}^3)$ for $r = 2$. On the other hand, for any $r > 2$, we show that the solution to the gNS can develop *norm inflation* in the sense that arbitrarily small initial data in the spaces $\dot{F}_{q_\alpha}^{-\alpha,r}(\mathbb{R}^3)$ can lead the corresponding solution to become arbitrarily large after an arbitrarily short time. In particular, such dichotomy of Triebel-Lizorkin spaces is also true for the classical N-S equations, i.e. $\alpha = 1$. Thus the Triebel-Lizorkin space framework naturally provides better connection between the well-known Koch-Tataru's BMO^{-1} well-posed work and Bourgain-Pavlović's $\dot{B}_{\infty}^{-1,\infty}$ ill-posed work.

Keywords: Generalized Navier-Stokes equations; Triebel-Lizorkin space; well-posedness; Ill-posedness.

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1 Introduction

In this article, we study the initial value problem of the following 3D generalized incompressible Navier-Stokes equations (gNS):

$$\begin{cases} \partial_t u + (-\Delta)^\alpha u + u \cdot \nabla u + \nabla p = 0, \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.1)$$

where $\alpha > \frac{1}{2}$, $(x, t) \in \mathbb{R}^3 \times (0, \infty)$, $u(x, t) = (u^1(x, t), u^2(x, t), u^3(x, t))$ are unknown vector functions, $p(x, t)$ is unknown scalar function, and $u_0(x)$ is a given vector function satisfying divergence free condition $\nabla \cdot u_0 = 0$.

Mathematical analysis of the classical incompressible Navier-Stokes (N-S) equations ($\alpha = 1$) has a long history. It goes back to Leray's famous work, i.e. [13], in

which Leray first introduced the concept of weak solutions and proved existence of global weak solutions associated with $L^2(\mathbb{R}^n)$ initial data by using an approximation approach and some weak compactness arguments. In 1964, Fujita-Kato [9] initiated a different approach and proved well-posedness of the initial value problem of the N-S in $H^s(\mathbb{R}^n)$ for $s \geq \frac{n}{2} - 1$. This approach was later extended to various other function spaces, see [3, 4, 12, 18] for expositions and references therein. An interesting result that must be mentioned is due to Koch-Tataru [11]. They proved that the solutions of N-S are well-posed in BMO^{-1} which is the largest function space for well-posedness. Besides these well-posedness results in critical spaces mentioned above, there also exist several works for super-critical initial value, for instance, existence of solutions for the initial value problem of the N-S for initial data in supercritical spaces $L^2(\mathbb{R}^n)$ space (which is supercritical for $n \geq 3$ but critical for $n = 2$) and sums of $L^2(\mathbb{R}^n)$ with some well-posedness spaces (cf. [2, 12]).

As we know, one crucial reason of working with the gNS equations on \mathbb{R}^3 for $\alpha > \frac{1}{2}$ is that they provides us the deeper understandings of the different actions of fractional Laplacian. Similar to the classical Navier-Stokes equations, one of the most primary problems is to establish local or global-in-time well-posedness of the gNS equation. Do the solutions exist in some spaces? If so, are they unique and is the system stable for certain initial data? By stable we mean that small perturbation of initial data guarantees small perturbation of solution. It is worth mentioning that either stability or instability of the nonlinear PDEs has a lot of applications in numerical analysis field.

Up to now, there exist many interesting works about the well-posedness and ill-posedness of gNS equations. In Table 1.1, we list several important progresses in the Besov space framework:

Well-posedness/ ill-posedness for the 3D generalized Navier-Stokes equations	
$\frac{1}{2} < \alpha < 1$	Well-posed in $\dot{B}_{\infty}^{1-2\alpha,\infty}$ [14, 23].
$\alpha = 1$	Well-posed in BMO^{-1} [11], Ill-posed in $\dot{B}_{\infty}^{-1,\infty}$ [1] and ill-posed in the logarithm-type Besov space [24].
$1 < \alpha < \frac{5}{4}$	Well-posed in $\dot{B}_2^{s,r}$ with $s = 1 - 2\alpha + \frac{3}{2}$ and $1 \leq r \leq \infty$ [21], Ill-posed in $\dot{B}_{\infty}^{1-2\alpha,\infty}$ [6].
$\frac{5}{4} \leq \alpha$	Global existence and uniqueness of classical solutions [15].

Table 1.1

Another result must be mentioned is that Cheskidov and Dai [5] proved norm-inflation of the gNS in subcritical Besov spaces $\dot{B}_{\infty}^{-\alpha,\infty}$ for $\alpha > 1$.

All the function spaces in Tabel 1.1 are *invariant under scaling* $u_{0\lambda} = \lambda^{2\alpha-1}u_0(\lambda x)$, which corresponds to the solution u of the gNS scale invariant under transformation

$$u_{\lambda}(x, t) = \lambda^{2\alpha-1}u(\lambda x, \lambda^{2\alpha}t) \quad \text{with } \lambda > 0 \text{ and } \alpha > 1/2. \quad (1.2)$$

By applying [12, Proposition 4.2] to (1.2), we can check that $\dot{B}_\infty^{1-2\alpha,\infty}$ is the largest scale invariant function space for the gNS equations. From Table 1.1, we observe that, when $\frac{1}{2} < \alpha < 1$, the gNS equations are well-posed in the largest scale invariant function space $\dot{B}_\infty^{1-2\alpha,\infty}$ ([14, 23]), but when $1 \leq \alpha < \frac{5}{4}$, the gNS equations actually are ill-posed in $\dot{B}_\infty^{1-2\alpha,\infty}$ ([1, 6]). Hence for the cases $1 \leq \alpha < \frac{5}{4}$, there exists certain difference between well-posed space with the largest scale invariant space $\dot{B}_\infty^{1-2\alpha,\infty}$.

In fact, when $\alpha = 1$ (i.e. the classical N-S equation), the best well-posed space is $\dot{F}_\infty^{-1,2} = BMO^{-1}$ given by Koch-Tataru [11]. Furthermore, $\dot{F}_\infty^{-1,2} \subsetneq \dot{B}_\infty^{-1,\infty}$ and there does exist a minor difference between $\dot{F}_\infty^{-1,2}$ and $\dot{B}_\infty^{-1,\infty}$. When $1 < \alpha < \frac{5}{4}$, the known results on well-posedness, for instance, $B_2^{s,r}$ with $1 \leq r \leq \infty$ and $s = 1 - 2\alpha + \frac{3}{2}$ (see [21]), are surely not the largest well-posed space of the gNS from (2.16) (see Subsection 2.3 below). Hence it would be very interesting to figure out what is the largest well-posed space and examine how large the difference is with $\dot{B}_\infty^{1-2\alpha,\infty}$. Roughly speaking, we guess the difference between the largest well-posed space and the largest scale invariant space $\dot{B}_\infty^{1-2\alpha,\infty}$ should enlarge as α increases.

In this paper, we investigate the interesting problem in critical Triebel-Lizorkin space framework and obtain the following conclusions about the 3D gNS with $1 \leq \alpha < \frac{5}{4}$ in the whole space \mathbb{R}^3 :

Well-posedness and ill-posedness of 3D gNS in critical Triebel-Lizorkin spaces	
$\alpha = 1$	well-posed in $\dot{F}_\infty^{-1,2}$ ([11]) and ill-posed in $\dot{F}_\infty^{-1,r>2}$ (Remark 1.4)
$1 < \alpha < \frac{5}{4}$	well-posed in $\dot{F}_{q_\alpha}^{-\alpha,2}$ and ill-posed in $\dot{F}_{q_\alpha}^{-\alpha,r>2}$ (Theorems 1.2–1.3)

Table 1.2

Table 1.2 shows that the largest Triebel-Lizorkin-type well-posed space for gNS is $\dot{F}_{q_\alpha}^{-\alpha,2}$ for any $1 \leq \alpha < \frac{5}{4}$. Indeed, for any $r < 2$, we have $\dot{F}_{q_\alpha}^{-\alpha,r} \subsetneq \dot{F}_{q_\alpha}^{-\alpha,2}$ and for any $r > 2$, we prove that the gNS is ill-posed in $\dot{F}_{q_\alpha}^{-\alpha,r}(\mathbb{R}^3)$ in the sense that arbitrarily small initial data can lead the corresponding solution to become arbitrarily large after an arbitrarily short time. Hence the results in Table 1.2 sharpen the well-posed analysis of gNS in the critical Triebel-Lizorkin space $\dot{F}_{q_\alpha}^{-\alpha,2}$ for all $\alpha \in [1, \frac{5}{4})$.

The results obtained here and [5] indicate that ill-posedness is more closely related to the smoothing indices s than the integrability indices q . More precisely, the gNS is ill-posed in $\dot{B}_q^{-\alpha,\infty}$ for $q_\alpha \leq q \leq \infty$.

Next we would like to introduce the *ideas* of the paper. In most applications, people use space-time type norm where one takes the space norm first. Here, we use some time-space type norm. This seems to be critical since it seems impossible to get the best results without these new norms and the related estimates in $L_x^{q_\alpha} L_T^2$ in which we have an equivalent characterization of Triebel-Lizorkin space by fractional semigroup (see Appendix A below). Then combining the a-priori bilinear estimates and the contraction arguments we prove the well-posedness. To show the

ill-posedness, we shall adopt the novel framework of norm inflation first introduced by Bourgain-Pavlović [1] in their study of the ill-posedness of the Navier–Stokes equation in $\dot{B}_{\infty}^{-1,\infty}(\mathbb{T}^3)$; but in doing so, we introduce some new inputs to the gNS. In particular, we make use of Hardy-Littlewood maximal function to estimate the norm of the solution in Triebel-Lizorkin space. Finally, we *conjecture* that for the specially constructed initial data in Subsection 3.2 below, there should exist a unique global classical solution since the data is not only energy finite but also essentially 2-dimensional/ summation of plane waves.

In order to prove the main results, we first recall the definition of homogeneous Besov spaces and Triebel-Lizorkin spaces ([20]). Let $\varphi(\xi) = \varphi(|\xi|)$ be a real-valued smooth function such that $0 \leq \varphi(\xi) \leq 1$ and

$$\text{supp } \varphi \subset \{\xi \in \mathbb{R}^3; 3/4 \leq |\xi| \leq 8/3\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for any } \xi \neq 0. \quad (1.3)$$

For any tempered distribution f and $i, j \in \mathbb{Z}$, we define the dyadic block as follows:

$$\Delta_j f(x) = \varphi(2^{-j}\nabla)f(x) \quad \text{and} \quad \Delta_i \Delta_j f \equiv 0 \quad \text{if } |i - j| \geq 2. \quad (1.4)$$

To exclude nonzero polynomials in homogeneous Besov spaces and Triebel-Lizorkin spaces, it is natural to use $Z'(\mathbb{R}^3)$ to denote the subspace of tempered distribution $f \in \mathcal{S}'(\mathbb{R}^3)$ modulo all polynomials set $P(\mathbb{R}^3)$, i.e.

$$Z'(\mathbb{R}^3) = \mathcal{S}'(\mathbb{R}^3)/P(\mathbb{R}^3).$$

Now we give the definition of Besov and Triebel-Lizorkin spaces, see [20].

Definition 1.1. For any $(s, q, r) \in (-\infty, \infty) \times [1, \infty] \times [1, \infty]$, we denote by $\dot{B}_q^{s,r}(\mathbb{R}^3)$ the set of distribution $f \in Z'(\mathbb{R}^3)$ satisfying

$$\|f\|_{\dot{B}_q^{s,r}(\mathbb{R}^3)} = \|\{2^{js}\|\Delta_j f\|_{L_x^q}\}\|_{l^r} < \infty$$

and we let $\dot{F}_q^{s,r}(\mathbb{R}^3)$ with $1 \leq q < \infty$ be the set of distribution $f \in Z'(\mathbb{R}^3)$ satisfying

$$\|f\|_{\dot{F}_q^{s,r}(\mathbb{R}^3)} = \|\|\{2^{js}\Delta_j f\}\|_{l^r}\|_{L_x^q} < \infty.$$

In particular, $\dot{F}_{\infty}^{s,r}$ is defined by the usual modification as in [20, Definition, p.30].

We are ready to state our main results on well-posedness and ill-posedness.

Theorem 1.2. (Well-posedness) Let $1 < \alpha < \frac{5}{4}$ and $q_{\alpha} = \frac{3}{\alpha-1}$. Then for any $u_0 \in \dot{F}_{q_{\alpha}}^{-\alpha,2}(\mathbb{R}^3)$ satisfying $\nabla \cdot u_0 = 0$, there exists $T = T(u_0) > 0$ such that system (1.1) has a unique local solution u satisfying

$$u \in C([0, T]; \dot{F}_{q_{\alpha}}^{-\alpha,2}(\mathbb{R}^3)) \cap L_x^{q_{\alpha}} L_T^2.$$

Furthermore, if $\|u_0\|_{\dot{F}_{q_{\alpha}}^{-\alpha,2}(\mathbb{R}^3)}$ is small enough, then system (1.1) has a unique global solution satisfying

$$u \in C([0, \infty); \dot{F}_{q_{\alpha}}^{-\alpha,2}(\mathbb{R}^3)) \cap L_x^{q_{\alpha}} L_t^2.$$

Theorem 1.3. (Ill-posedness) *For any $1 < \alpha < \frac{5}{4}$, $r > 2$, $\delta > 0$ and $q_\alpha = \frac{3}{\alpha-1}$, there exists a solution u to system (1.1) with initial data $u_0 \in \dot{F}_{q_\alpha}^{-\alpha, r}(\mathbb{R}^3)$ satisfying*

$$\|u_0\|_{\dot{F}_{q_\alpha}^{-\alpha, r}(\mathbb{R}^3)} \lesssim \delta$$

and $\nabla \cdot u_0 = 0$ such that for some $0 < T < \delta$,

$$\|u(T)\|_{\dot{F}_{q_\alpha}^{-\alpha, r}(\mathbb{R}^3)} \gtrsim \frac{1}{\delta}.$$

Remark 1.4.

(i) In the proof of Theorem 1.3, the constructed initial datum satisfy several good properties: *real-valued, smooth, energy finite in the whole space, and essentially plane waves (almost $2D$).*

(ii) When $\alpha = 1$ and $q_\alpha = \infty$, $\dot{F}_\infty^{-1, 2} = BMO^{-1}$, by the Koch-Tataru's well-posed work [11] in BMO^{-1} , we also proved the *ill-posedness* of the Navier-Stokes equations in the Triebel-Lizorkin spaces $\dot{F}_\infty^{-1, r}(\mathbb{R}^3)$ (cf. [7]) for $2 < r < \infty$ which are strictly smaller than $\dot{B}_\infty^{-1, \infty}(\mathbb{R}^3)$ in which Bourgain and Pavlović proved ill-posedness of the Navier-Stokes equations, see [1].

This paper is *organized* as follows: In Section 2, we mainly establish the well-posedness by proving a key bilinear estimate on $L_x^{q_\alpha} L_T^2$. Moreover, we also consider many other bilinear estimates in the end of this section. In particular, by interpolation several applications of our bilinear estimate are given; In Section 3, we first construct a very special initial data and list some necessary remarks, and then we establish all the desired estimates about the first and second approximation terms which will be used in controlling the remainder term. Finally, combining all the a-priori estimates we prove ill-posedness of the gNS.

Notations: Throughout this paper, we shall use C and c to denote universal constants and may change from line to line. Both $\mathcal{F}f$ and \hat{f} stand for Fourier transform of f with respect to space variable, while \mathcal{F}^{-1} stands for the inverse Fourier transform. We denote $A \leq CB$ by $A \lesssim B$ and $A \lesssim B \lesssim A$ by $A \sim B$. For any $1 \leq p \leq \infty$, we denote $L^p(0, T)$, $L^p(T_1, T_2)$, $L^p(0, \infty)$ and $L^q(\mathbb{R}^3)$ by L_T^p , $L_{[T_1, T_2]}^p$, L_t^p and L_x^q , respectively. Later on, we use $\dot{F}_q^{s, r}$ to denote $\dot{F}_q^{s, r}(\mathbb{R}^3)$ if there is no confusion about the domain, and similar conventions are applied.

2 Analysis of well-posedness

In this section, we will prove well-posedness of the 3D gNS in $\dot{F}_{q_\alpha}^{-\alpha, 2}$ for $q_\alpha = \frac{3}{\alpha-1}$ and $1 < \alpha < \frac{5}{4}$. Notice that for $\alpha = 1$, $\dot{F}_{q_\alpha}^{-\alpha, 2} = BMO^{-1}$ in which the well-posedness is proved by Koch and Tataru, see [11].

As usual, we first write (1.1) into the following equivalent mild integral equations:

$$u = e^{-t(-\Delta)^\alpha} u_0 - \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P}(u \cdot \nabla v) d\tau, \quad (2.1)$$

where \mathbb{P} is the Leray projection operator and $\mathbb{P} = Id - \nabla \frac{1}{\Delta} \operatorname{div}$.

For simplicity, we denote the bilinear term by

$$B(u, v) := \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P}(u \cdot \nabla v) d\tau. \quad (2.2)$$

To prove well-posedness, we first prove several preliminary lemmas including the endpoint bilinear estimate. Based on these estimates, the well-posedness immediately follows from the standard Picard iteration principle. Finally, we also give some other bilinear estimates and their applications.

2.1 Preliminaries

In this subsection, we first give several preliminary lemmas. The first lemma is about the point-wise estimates for the kernel of fractional semigroup $e^{-t(-\Delta)^\alpha}$ with a regularized operator $(-\Delta)^{s/2}$. For convenience of other applications, we consider dimension $n \geq 2$ and also allow $\alpha > 0$. For any $s > -n$, $j \in \mathbb{Z}$, $t > 0$ and $x \in \mathbb{R}^n$, let

$$K_s^\alpha(t, x) = \mathcal{F}^{-1} \left(|\xi|^s e^{-t|\xi|^{2\alpha}} \right) (x),$$

and

$$K_{s,j}^\alpha(t, x) = \mathcal{F}^{-1} \left(|\xi|^s e^{-t|\xi|^{2\alpha}} \varphi(2^{-j}\xi) \right) (x).$$

where $\varphi(\xi)$ be a smoothing truncation function defined in (1.3). Clearly, they are the kernel of the operator families $(-\Delta)^{s/2} e^{-t(-\Delta)^\alpha}$ and $(-\Delta)^{s/2} e^{-t(-\Delta)^\alpha} \Delta_j$, respectively. As $s = 0$, we denote the kernels by $K^\alpha(t, x)$ and $K_j^\alpha(t, x)$, respectively.

Lemma 2.1. *Let $\alpha > 0$, $n \geq 2$, $s > -n$ and $\beta \in \mathbb{N}^n$. Then we have the following estimates:*

$$|\partial_x^\beta K_s^\alpha(t, x)| \leq \begin{cases} C (t^{\frac{1}{2\alpha}} + |x|)^{-(n+s+|\beta|)}, & s \neq 0; \\ C t(t^{\frac{1}{2\alpha}} + |x|)^{-(n+2|\alpha|+|\beta|)}, & s = 0. \end{cases} \quad (2.3)$$

and for any $N \geq 1$,

$$|\partial_x^\beta K_{s,j}^\alpha(t, x)| \leq C_N e^{-ct2^{2j\alpha}} 2^{j(n+s+|\beta|)} (1 + |2^j x|)^{-N}. \quad (2.4)$$

Proof. For the first estimates (2.3), it seems to be well-known. For instance, one can see [12, Proposition 11.1], [8, Remark 2.2] and [14, 17] for the details of the proof.

In order to prove the (2.4), recall the definitions of $K_{s,j}^\alpha(t, x)$, by scaling we obtain that

$$\begin{aligned} \left| \partial_x^\beta K_{s,j}^\alpha(t, x) \right| &= \left| \int_{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}} e^{i\xi \cdot 2^j x} e^{-t2^{2\alpha j} |\xi|^\alpha} 2^{(s+n+|\beta|)j} \xi^\beta |\xi|^s \varphi(\xi) d\xi \right| \\ &\leq C 2^{j(s+n+|\beta|)} e^{-ct2^{2\alpha j}} \end{aligned} \quad (2.5)$$

and for any $N \geq 1$,

$$\begin{aligned} &|2^j x|^N \left| \partial_x^\beta K_{s,j}^\alpha(t, x) \right| \\ &\leq C_N 2^{j(s+n+|\beta|)} \sum_{|\mu|=N} \left| \int_{\mathbb{R}^n} e^{i\xi \cdot 2^j x} \partial_\xi^\mu (e^{-t2^{2\alpha j} |\xi|^{2\alpha}} \xi^\beta |\xi|^s \varphi(\xi)) d\xi \right| \\ &\leq C'_N 2^{j(s+n+|\beta|)} \int_{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}} e^{-t2^{2\alpha j} |\xi|^{2\alpha}} (t^N 2^{2\alpha N j} + 1) \left(\sum_{|\mu| \leq N} |\partial_\xi^\mu \varphi(\xi)| \right) d\xi \\ &\leq C''_N 2^{j(s+n+|\beta|)} e^{-ct2^{2\alpha j}}, \end{aligned} \quad (2.6)$$

where we use the bound $e^{-t2^{2\alpha j} |\xi|^{2\alpha}} (t^N 2^{2\alpha N j} + 1) \leq C_N e^{-ct2^{2\alpha j}}$ as $\frac{3}{4} \leq |\xi| \leq \frac{8}{3}$. Combining (2.5) and (2.6), we prove (2.4). \square

The following *endpoint bilinear estimate* follows by using Lemma 2.1 and the classical Hardy-Littlewood-Sobolev (H-L-S) inequality (see [19, p. 353]).

Lemma 2.2. *Let $B(v, w)$ be defined as in (2.2), $1 < \alpha < \frac{5}{4}$ and $q_\alpha = \frac{3}{\alpha-1}$. Then for any $T > 0$, there exists a positive constant C_α depending only on α such that*

$$\|B(v, w)\|_{L_x^{q_\alpha} L_T^2 \cap L_T^\infty \dot{F}_{q_\alpha}^{-\alpha, 2}} \leq C_\alpha \|v\|_{L_x^{q_\alpha} L_T^2} \|w\|_{L_x^{q_\alpha} L_T^2}. \quad (2.7)$$

Proof. To prove (2.7), by applying Lemma 2.1, Young's inequality with respect to time variable, and H-L-S inequality with $1 < \frac{q_\alpha}{2} < \infty$, $\frac{-1+\alpha}{3} = \frac{2}{q_\alpha} - \frac{1}{q_\alpha}$ to $B(v, w)$, we have

$$\begin{aligned} \|B(v, w)\|_{L_x^{q_\alpha} L_T^2} &= \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (v \otimes w) d\tau \right\|_{L_x^{q_\alpha} L_T^2} \\ &\lesssim \left\| \int_{\mathbb{R}^3} \int_0^t \left((t-\tau)^{\frac{1}{2\alpha}} + |x-y| \right)^{-4} |(v \otimes w)(y, \tau)| d\tau dy \right\|_{L_x^{q_\alpha} L_T^2} \\ &\lesssim \left\| \int_{\mathbb{R}^3} |x-y|^{-4+\alpha} \|(v \otimes w)(y, \cdot)\|_{L_T^1} dy \right\|_{L_x^{q_\alpha}} \\ &\lesssim \|v \otimes w\|_{L_x^{\frac{q_\alpha}{2}} L_T^1}. \end{aligned}$$

Next, using the boundedness of \mathbb{P} in homogeneous space $\dot{F}_{q_\alpha}^{-\alpha, 2}$, $\dot{F}_{q_\alpha}^{0, 2} = L_x^{q_\alpha}$ and Minkowski inequality, we get

$$\|B(v, w)\|_{L_T^\infty \dot{F}_{q_\alpha}^{-\alpha, 2}} \lesssim \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} (v \otimes w) d\tau \right\|_{L_T^\infty \dot{F}_{q_\alpha}^{1-\alpha, 2}}$$

$$\begin{aligned}
&\lesssim \left\| \int_0^t \left(K_{1-\alpha}^\alpha(t-\tau, \cdot) * ((v \otimes w)(\tau, \cdot)) \right)(x) d\tau \right\|_{L_T^\infty L_x^{q_\alpha}} \\
&\lesssim \left\| \int_0^t \left(K_{1-\alpha}^\alpha(t-\tau, \cdot) * ((v \otimes w)(\tau, \cdot)) \right)(x) d\tau \right\|_{L_x^{q_\alpha} L_T^\infty}, \quad (2.8)
\end{aligned}$$

where $K_{1-\alpha}^\alpha(t, x)$ is the kernel of $e^{t\Delta}(-\Delta)^{\frac{1-\alpha}{2}}$. Making use of Lemma 2.1, we have

$$\sup_{t>0} |K_{1-\alpha}^\alpha(t, x-y)| \lesssim |x-y|^{-4+\alpha}. \quad (2.9)$$

Plugging (2.9) into (2.8), using the H-L-S inequality and Hölder inequality, we get

$$\begin{aligned}
(2.8) &\lesssim \left\| \int_{\mathbb{R}^3} \frac{1}{|x-y|^{3-\alpha+1}} \|v \otimes w(y, \cdot)\|_{L_T^1} dy \right\|_{L_x^{q_\alpha}} \\
&\lesssim \left\| \|v \otimes w\|_{L_T^1} \right\|_{L_x^{\frac{q_\alpha}{2}}} \lesssim \|v\|_{L_x^{q_\alpha} L_T^2} \|w\|_{L_x^{q_\alpha} L_T^2}.
\end{aligned}$$

Hence we finish the proof of (2.7). \square

The next lemma is about the equivalent definition and characterization of the Triebel-Lizorkin space which proof is given in the Appendix.

Lemma 2.3. *For any $1 < \alpha < \frac{5}{4}$ and $q_\alpha = \frac{3}{\alpha-1}$, we have the following equivalent definition of $\dot{F}_{q_\alpha}^{-\alpha, 2}$, i.e. any $f \in \dot{F}_{q_\alpha}^{-\alpha, 2}$ satisfy*

$$\|f\|_{\dot{F}_{q_\alpha}^{-\alpha, 2}} \approx \left(\int_{\mathbb{R}^3} \left(\int_0^\infty |e^{-t(-\Delta)^\alpha} f|^2 dt \right)^{\frac{q_\alpha}{2}} dx \right)^{1/q_\alpha}. \quad (2.10)$$

Remark 2.4.

(i) From (2.10), we observe that given $u_0 \in \dot{F}_{q_\alpha}^{-\alpha, 2}$ and for any $\varepsilon > 0$, there exists positive $T = T(u_0, \varepsilon)$ depending on the profile of u_0 such that

$$\|e^{-t(-\Delta)^\alpha} u_0\|_{L_x^{q_\alpha} L_T^2} \leq \varepsilon.$$

One way to prove it is using splitting method or approximation by good functions.

(ii) When $\alpha = 1$, $\dot{F}_\infty^{-1, 2} = BMO^{-1}$ and it has the following equivalent Carleson measure characterization which is closely related to (2.10) (see [11]):

$$\|f\|_{\dot{F}_\infty^{-1, 2}} \approx \sup_{x \in \mathbb{R}^3, R > 0} \left(\frac{1}{B_R(x)} \int_0^{R^2} \int_{B_R(x)} |e^{t\Delta} f|^2 dt dx \right)^{\frac{1}{2}}.$$

2.2 The proof of Theorem 1.2

In order to prove well-posedness of the 3D gNS, we need to use the following Picard contraction principle, see for instance, [4, Chapter 3.1, Lemma 4] and [12].

Lemma 2.5. *Let $(X, \|\cdot\|_X)$ be an abstract Banach space and $B : X \times X \rightarrow X$ be a bilinear operator. If for any $(u, v) \in X \times X$, there exists $c > 0$ such that*

$$\|B(u, v)\|_X \leq c\|u\|_X\|v\|_X,$$

then for any u_0 satisfying $\|e^{t\Delta}u_0\|_X < 1/4c$, the following system $u = e^{t\Delta}u_0 + B(u, u)$ has a solution u in X . In particular, the solution is such that $\|u\|_X \leq 2\|e^{t\Delta}u_0\|_X$ and is the only one such that $\|u\|_X < 1/2c$.

Now we are ready to prove the local and global well-posedness.

Proof of local well-posedness: Using Lemmas 2.2, 2.3 and 2.5, we prove that there exists a unique solution u in a closed ball in $L_x^{q\alpha} L_T^2$ since $\|e^{-t(-\Delta)^\alpha}u_0\|_{L_x^{q\alpha} L_T^2} < \frac{1}{4c}$ if $u_0 \in \dot{F}_{q\alpha}^{-\alpha, 2}$ and T is small enough. Next we prove additional property of u via Lemma 2.2, i.e. $u \in L_T^\infty \dot{F}_{q\alpha}^{-\alpha, 2}$. At last, following the standard dense argument we show that $u \in C([0, T]; \dot{F}_{q\alpha}^{-\alpha, 2})$, see [10] for details of Picard iteration arguments.

Proof of global well-posedness: Noticing that bilinear estimates in Lemma 2.2 can be extended to $T = \infty$. However, in this case, $\|e^{-t(-\Delta)^\alpha}u_0\|_{L_x^{q\alpha} L_t^2}$ is not necessarily small. Hence smallness condition is needed for global well-posedness. Proof of global well-posedness follows in the similar way.

2.3 Applications of other bilinear estimates

In this section, we consider many other bilinear estimates and give several applications to the analysis of well-posedness for the 3D gNS. Moreover, it is worth mentioning that these bilinear estimates are also very important to the ill-posedness of the 3D gNS.

At first, we recall the endpoint bilinear estimates proved in Lemma 2.2:

$$\|B(u, v)\|_{L_x^{q\alpha} L_T^2} = \left\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} \mathbb{P} \nabla \cdot (u \otimes v) d\tau \right\|_{L_x^{q\alpha} L_T^2} \lesssim \|u\| \|v\|_{L_x^{\frac{q\alpha}{2}} L_T^1}. \quad (2.11)$$

It is clear that $L_x^{\frac{3}{2\alpha-1}} L_T^\infty$ is another endpoint space-time space for the 3D gNS equations. This kind of space-times space was first introduced by Calderón [2] to study the incompressible Navier-Stokes equations. Applying maximal function theory and H-L-S inequality, we get $\|\sup_{t>0} |e^{-t(-\Delta)^\alpha}u_0|\|_{L_x^{\frac{3}{2\alpha-1}}} \lesssim \|u_0\|_{L_x^{\frac{3}{2\alpha-1}}}$ and

$$\|B(u, v)\|_{L_x^{\frac{3}{2\alpha-1}} L_T^\infty} \lesssim \|(-\Delta)^{\frac{1-2\alpha}{3}}(\|u\| \|v\|_{L_T^\infty})\|_{L_x^{\frac{3}{2\alpha-1}}} \lesssim \|u\| \|v\|_{L_x^{\frac{3}{4\alpha-2}} L_T^\infty}. \quad (2.12)$$

Hence *local and global well-posedness of gNS for small $L^{\frac{3}{2\alpha-1}}(\mathbb{R}^3)$ data* follows from Picard contraction argument. Additionally, *local well-posedness of the 3D gNS for large $L^{\frac{3}{2\alpha-1}}(\mathbb{R}^3)$ data* follows in the similar way as in [10].

For any $\frac{1}{q_\theta} = \frac{\theta}{q_\alpha} + \frac{(1-\theta)(2\alpha-1)}{3}$, $\frac{1}{p_\theta} = \frac{\theta}{2} + \frac{1-\theta}{\infty}$ and $0 < \theta < 1$, by Interpolating (2.11)–(2.12), we have

$$\|B(u, v)\|_{L_x^{q_\theta} L_T^{p_\theta}} \lesssim \| |u| |v| \|_{L_x^{\frac{q_\theta}{2}} L_T^{\frac{p_\theta}{2}}}. \quad (2.13)$$

Thus (2.13) and (A.6) yield *well-posedness of the gNS for $\dot{F}_{q_\theta}^{-\theta\alpha, p_\theta}(\mathbb{R}^3)$ data*.

Notice that by applying the similar arguments as introduced by Fujita-Kato (cf. [9]), one can prove local and global well-posedness for any $\dot{H}^{\frac{5-4\alpha}{2}}(\mathbb{R}^3)$ initial data. Thus interpolation between $\dot{F}_{q_\alpha}^{-\alpha, 2}(\mathbb{R}^3)$ and $\dot{F}_2^{\frac{5-4\alpha}{2}, 2}(\mathbb{R}^3)$ yields *well-posedness* of the gNS in any Triebel-Lizorkin space $\dot{F}_{q_\theta}^{s_\theta, 2}(\mathbb{R}^3)$ with $s_\theta = \theta(-\alpha) + \frac{(1-\theta)(5-4\alpha)}{2}$, $\frac{1}{q_\theta} = \frac{\theta}{q_\alpha} + \frac{1-\theta}{2}$ and $0 < \theta < 1$ (see [20, p. 44]).

Recall the ideas that Cannone and Planchon used to prove the well-posedness of the N-S equations for data in the Besov space $B_q^{s, r}(\mathbb{R}^n)$. By making use of Lemma 2.2, it is easy to prove that for any $\frac{3}{2\alpha-1} < q < \frac{3}{\alpha-1}$ and $1 < \alpha < \frac{5}{4}$,

$$\sup_{t>0} t^{\frac{2\alpha-1-\frac{3}{q}}{2\alpha}} \|B(v, w)\|_{L_x^q} \lesssim \sup_{\tau>0} \left(\tau^{\frac{2\alpha-1-\frac{3}{q}}{2\alpha}} \|v\|_{L_x^q} \right) \sup_{\tau>0} \left(\tau^{\frac{2\alpha-1-\frac{3}{q}}{2\alpha}} \|w\|_{L_x^q} \right) \quad (2.14)$$

Moreover, from [17, Proposition 2.1], we have

$$\sup_{t>0} t^{\frac{2\alpha-1-\frac{3}{q}}{2\alpha}} \|e^{-t(-\Delta)^\alpha} u_0\|_{L_x^q} \sim \|u_0\|_{\dot{B}_q^{1-2\alpha+\frac{3}{q}, \infty}}. \quad (2.15)$$

Combining (2.14)–(2.15), we can prove well-posedness of the gNS in $\dot{B}_q^{1-2\alpha+\frac{3}{q}, \infty}(\mathbb{R}^3)$ with $\frac{3}{2\alpha-1} < q < q_\alpha = \frac{3}{\alpha-1}$. Moreover, we have

$$\dot{B}_2^{1-2\alpha+\frac{3}{2}, r \geq 1}(\mathbb{R}^3) \hookrightarrow \dot{B}_q^{1-2\alpha+\frac{3}{q}, \infty}(\mathbb{R}^3). \quad (2.16)$$

Next, we shall consider the bicontinuity of $B(u, v)$ in some spaces which are not scale invariant. For instance, we prove the following bilinear estimate:

$$\begin{aligned} \|B(u, v)\|_{L_x^\infty L_T^2} &\lesssim \left\| \int_{\mathbb{R}^3} \int_0^t \frac{|u(\tau, y)| |v(\tau, y)|}{((t-\tau)^{\frac{1}{2\alpha}} + |x-y|)^{n+1}} d\tau dy \right\|_{L_x^\infty L_T^2} \\ &\lesssim T^{\frac{1}{2}} \left\| \int_{|x-y| \geq 1} \frac{\| |u| |v| \|_{L_T^1}}{|x-y|^{n+1}} dy \right\|_{L_x^\infty} + \left\| \int_{|x-y| \leq 1} \frac{\| |u| |v| \|_{L_T^1}}{|x-y|^{n-\alpha+1}} dy \right\|_{L_x^\infty} \\ &\lesssim \max\{T^{\frac{1}{2}}, 1\} \left\| |u| |v| \right\|_{L_x^\infty L_T^1}. \end{aligned} \quad (2.17)$$

Interpolating (2.11) and (2.17), then using (A.6), for any $q_\alpha < q < \infty$ and $T \lesssim 1$, we have

$$\|B(u, v)\|_{L_x^q L_T^2} \lesssim \|u\|_{L_x^{\frac{q}{2}} L_T^1} \|v\|_{L_x^{\frac{q}{2}} L_T^1}, \quad \|e^{-t(-\Delta)^\alpha} u_0\|_{L_x^q L_t^2} \sim \|u_0\|_{\dot{F}_q^{-\alpha, 2}}, \quad (2.18)$$

which yields *local well-posedness of the 3D gNS with $\dot{F}_q^{-\alpha, 2}$ -valued initial data*.

3 Analysis of ill-posedness

In this section, we will prove “norm inflation” of the gNS in $\dot{F}_{q_\alpha}^{-\alpha, r}$ with $r > 2$ and $q_\alpha = \frac{3}{\alpha-1}$. Following the ideas in [1], we rewrite the solution to the gNS equations as a summation of the first approximation terms, the second approximation terms and remainder terms, i.e.

$$u = u_1 - u_2 + y, \quad (3.1)$$

where $u_1 := e^{-t(-\Delta)^\alpha} u_0 := S_t u_0$ and $u_2 = B(u_1, u_1)$. Moreover, the remainder terms satisfy the following integral equations:

$$y = G_0 + G_1 - G_2, \quad (3.2)$$

on $(0, \infty)$ with the initial conditions $y(0) = 0$,

$$\begin{cases} G_2 = B(y, y), \\ G_1 = B(y, u_2) + B(u_2, y) - B(y, u_1) - B(u_1, y), \\ G_0 = B(u_2, u_1) + B(u_1, u_2) - B(u_2, u_2). \end{cases} \quad (3.3)$$

In the rest part of this section, we will establish the *a-priori* estimates for u_0 , u_1 , u_2 and y . Precisely, in Subsection 3.1, we construct some special initial data u_0 ; In Subsection 3.2 we estimate the small upper bounds of u_0 and u_1 ; In Subsection 3.3, we prove both upper bound and lower bound of u_2 ; In Subsection 3.4, we prove the upper bound of y ; In Subsection 3.5, we complete the proof of Theorem 1.3.

3.1 Construction of initial data for the 3D gNS equations

For any fixed small number $\delta > 0$, we define the initial data as follows:

$$\begin{aligned} u_0 &= \frac{Q}{\sqrt{\rho}} \sum_{s=1}^{\rho} |k_s|^\alpha \left\{ \begin{pmatrix} 0 \\ \frac{-\partial_3}{|k_s|} \psi_{k_s} \\ \frac{\partial_2}{|k_s|} \psi_{k_s} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{-\partial_3}{|k_s|} \bar{\psi}_{k_s} \\ \frac{\partial_2}{|k_s|} \bar{\psi}_{k_s} \end{pmatrix} + \begin{pmatrix} \frac{\partial_2}{|k_s|} \psi_{k'_s} \\ \frac{-\partial_1}{|k_s|} \psi_{k'_s} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial_2}{|k_s|} \bar{\psi}_{k'_s} \\ \frac{-\partial_1}{|k_s|} \bar{\psi}_{k'_s} \\ 0 \end{pmatrix} \right\} \\ &:= \frac{Q}{\sqrt{\rho}} \sum_{s=1}^{\rho} |k_s|^\alpha (\Psi_{k_s} + \bar{\Psi}_{k_s} + \Phi_{k'_s} + \bar{\Phi}_{k'_s}) \end{aligned} \quad (3.4)$$

with $\bar{\Psi}_{k_s}$, $\bar{\Phi}_{k'_s}$, $\bar{\psi}_{k_s}$ and $\bar{\psi}_{k'_s}$ being conjugate functions of Ψ_{k_s} , $\Phi_{k'_s}$, ψ_{k_s} and $\psi_{k'_s}$, respectively. In addition, the parameters and auxiliary functions satisfy:

(H1) Q , ρ and m_0 will be chosen sufficiently large according to the size of δ and

$$k_0 = (0, 2^{m_0}, 0), \quad k_s = (0, 2^{\frac{(s+1)(s+2m_0)}{2}}, 0), \quad k'_s = (7, -2^{\frac{(s+1)(s+2m_0)}{2}}, 0),$$

with $s = 1, 2, \dots$ and ρ will be specified in Lemma 3.11.

(H2) $\psi(x)$, $\psi_{k_s}(x)$, $\psi_{k'_s}(x)$, $\bar{\psi}_{k_s}(x)$ and $\bar{\psi}_{k'_s}(x)$ satisfy:

$$\begin{aligned} \widehat{\psi}(\xi) &= \widehat{\psi}(|\xi|) \geq 0, \quad \text{supp } \widehat{\psi} \subset B_{\frac{1}{4}}(0) := \{|\xi| < \frac{1}{4}\}, \\ \|\widehat{\psi}\|_{L^1_\xi} &= 1, \quad \|\widehat{\psi}\|_{L^\infty_\xi} \sim \|\psi\|_{L^{q\alpha}_x} \sim 1, \quad \widehat{\psi}_{k_s}(\xi) = \widehat{\psi}(\xi - k_s), \\ \widehat{\psi}_{k'_s}(\xi) &= \widehat{\psi}(\xi - k'_s), \quad \widehat{\bar{\psi}}_{k_s}(\xi) = \widehat{\bar{\psi}}_{-k_s}(\xi), \quad \widehat{\bar{\psi}}_{k'_s}(\xi) = \widehat{\bar{\psi}}_{-k'_s}(\xi). \end{aligned}$$

Remark 3.1. From hypothesis (H1)–(H2), we have the following observations:

- i) $\psi(x)$ is real-valued and smooth, $\psi_{k_s}(x) = e^{ik_s \cdot x} \psi(x)$, $\bar{\psi}_{k_s}(x) = e^{i(-k_s) \cdot x} \psi(x) = \psi_{-k_s}(x)$. As a consequence, $u_0(x)$ is real-valued, smooth and divergence free.
- ii) For any $(k, q) \in \mathbb{Z}^3 \times [1, \infty]$, we have $\|\widehat{\psi}_k\|_{L^q_\xi} = \|\widehat{\psi}\|_{L^q_\xi}$ since L^q_ξ is a shift invariant space. Making use of Hausdorff–Young’s inequality and (H2), we get

$$\|\Psi_{k_s}\|_{L^{q\alpha}_x} + \|\bar{\Psi}_{k_s}\|_{L^{q\alpha}_x} + \|\Phi_{k'_s}\|_{L^{q\alpha}_x} + \|\bar{\Phi}_{k'_s}\|_{L^{q\alpha}_x} \lesssim \|\widehat{\psi}\|_{L^{\frac{q\alpha}{q\alpha-1}}_\xi} \lesssim 1. \quad (3.5)$$

- iii) *Lacunarity of the sequence $\{|k_s|\}_{s=1}^\rho$.* According to the choices of k_s , k'_s and m_0 in (H1), $\log_2 |k_s| = \frac{(s+1)(s+2m_0)}{2}$ is integer and $|k_s| < |k'_s| < |k_s| + 7$. For any positive integer $s \in [1, \rho] \cap \mathbb{N}$, if we denote $j_s = \log_2 |k_s| - 1$, then $j_{s+1} - j_s \geq m_0$. For sufficiently large m_0 (≥ 5), we have $7 < \frac{1}{4} 2^{m_0} = \frac{1}{4} |k_0| < \frac{1}{4} |k_s|$,

$$\frac{3}{4} 2^{j_s} < |k_s| < |k'_s| < \frac{8}{3} 2^{j_s}, \quad \frac{3}{4} 2^{j_s+1} < |k_s| < |k'_s| < \frac{8}{3} 2^{j_s+1}. \quad (3.6)$$

For suitably large m_0 and $j \leq j_s - 1$ or $j \geq j_s + 2$, from (3.6) we have

$$\left(B_{\frac{1}{4}}(k_s) \cup B_{\frac{1}{4}}(k'_s) \right) \cap \left\{ \xi \in \mathbb{R}^3; \frac{3}{4} 2^j < |\xi| < \frac{8}{3} 2^j \right\} = \emptyset. \quad (3.7)$$

From (1.3)–(1.4) and (3.4)–(3.7), for any $\ell \in \{0, 1\}$ and $j_i \in \{j_1, \dots, j_\rho\}$ we get

$$\Delta_{j_i+\ell} \sum_{s=1}^\rho \Psi_{k_s} = \Delta_{j_i+\ell} \Psi_{k_i}, \quad \Delta_{j_i+\ell} \sum_{s=1}^\rho \Phi_{k'_s} = \Delta_{j_i+\ell} \Phi_{k'_i}. \quad (3.8)$$

Moreover, for any $j \in \mathbb{Z} \setminus \{j_1, j_1 + 1, j_2, j_2 + 1, \dots, j_\rho, j_\rho + 1\}$, from (3.7) we have

$$\Delta_j \sum_{s=1}^\rho \Psi_{k_s} = \sum_{s=1}^\rho \Delta_j \Psi_{k_s} \equiv 0, \quad \Delta_j \sum_{s=1}^\rho \Phi_{k'_s} = \sum_{s=1}^\rho \Delta_j \Phi_{k'_s} \equiv 0. \quad (3.9)$$

3.2 Estimates for initial data and the first approximation terms

In this subsection, we will estimate u_0 and $u_1 = e^{-t(-\Delta)^\alpha} u_0$.

Lemma 3.2. *For any initial u_0 defined in (3.4) and any $r \geq 2$ and $1 < \alpha < \frac{5}{4}$, we obtain that*

$$\|u_0\|_{\dot{F}_{q\alpha}^{-\alpha,r}} \lesssim Q\rho^{\frac{1}{r}-\frac{1}{2}}, \quad \|u_1\|_{\dot{F}_{q\alpha}^{-\alpha,r}} \lesssim Q\rho^{\frac{1}{r}-\frac{1}{2}} e^{-ct|k_0|^{2\alpha}}, \quad (3.10)$$

for some absolute constant $c > 0$.

Proof. We first deal with the cases $2 \leq r < \infty$. In view of the construction of u_0 , it suffices to bound $Q\rho^{-\frac{1}{2}} \sum_{s=1}^{\rho} |k_s|^\alpha \Psi_{k_s}$ and $Q\rho^{-\frac{1}{2}} \sum_{s=1}^{\rho} |k_s|^\alpha e^{-t(-\Delta)^\alpha} \Psi_{k_s}$. By Definition 1.1, (3.6)–(3.9) and $|k_s| \sim 2^{j_s}$, we obtain

$$\begin{aligned} \left\| \sum_{s=1}^{\rho} |k_s|^\alpha \Psi_{k_s} \right\|_{\dot{F}_{q\alpha}^{-\alpha,r}} &= \left\| \left\{ 2^{-\alpha j} \Delta_j \left(\sum_{s=1}^{\rho} |k_s|^\alpha \Psi_{k_s} \right) \right\}_{l^r} \right\|_{L_x^{q\alpha}} \\ &= \left\| \left(\sum_{i=1}^{\rho} \sum_{\ell=0,1} 2^{-j_i \alpha r} |k_i|^{\alpha r} |\Delta_{j_i+\ell} \Psi_{k_i}|^r \right)^{\frac{1}{r}} \right\|_{L_x^{q\alpha}} \\ &\lesssim \left\| \left(\sum_{i=1}^{\rho} \sum_{\ell=0,1} |\Delta_{j_i+\ell} \Psi_{k_i}|^r \right)^{\frac{1}{r}} \right\|_{L_x^{q\alpha}}. \end{aligned} \quad (3.11)$$

Note that $|\psi_{k_i}(x)| = |\psi(x)|$ for any $1 \leq i \leq \rho$, then for any $\ell = 0, 1$ and $1 \leq i \leq \rho$, we have the following point-wise estimates

$$|\Delta_{j_i+\ell} \Psi_{k_i}| \leq \sum_{|\beta|=1} |(|k_i|^{-1} \partial^\beta \Delta_{j_i+\ell}) \psi_{k_i}| \lesssim M\psi, \quad (3.12)$$

where Mf denotes the standard Hardy-Littlewood maximal function of f . Hence it follows from (3.12), Hardy-Littlewood theorem [19, Chapter 1, p. 13] and $\|M\psi\|_{L_x^{q\alpha}} \lesssim \|\psi\|_{L_x^{q\alpha}} \lesssim 1$ that

$$(3.11) \lesssim \left\| \left(\sum_{i=1}^{\rho} \sum_{\ell=0,1} |M\psi|^r \right)^{\frac{1}{r}} \right\|_{L_x^{q\alpha}} \lesssim \rho^{\frac{1}{r}} \|M\psi\|_{L_x^{q\alpha}} \lesssim \rho^{\frac{1}{r}}, \quad (3.13)$$

which immediately concludes the desired estimates $\|u_0\|_{\dot{F}_{q\alpha}^{-\alpha,r}} \lesssim Q\rho^{\frac{1}{r}-\frac{1}{2}}$.

Next to estimate $\|u_1\|_{\dot{F}_{q\alpha}^{-\alpha,r}}$, similarly as in (3.11), we can obtain that

$$\left\| \sum_{s=1}^{\rho} |k_s|^\alpha e^{-t(-\Delta)^\alpha} \Psi_{k_s} \right\|_{\dot{F}_{q\alpha}^{-\alpha,r}} \lesssim \left\| \left(\sum_{i=1}^{\rho} \sum_{\ell=0,1} |\Delta_{j_i+\ell} e^{-t(-\Delta)^\alpha} \Psi_{k_i}|^r \right)^{\frac{1}{r}} \right\|_{L_x^{q\alpha}}. \quad (3.14)$$

Note that for $\ell = 0, 1$ and $1 \leq i \leq \rho$,

$$|\Delta_{j_i+\ell} e^{-t(-\Delta)^\alpha} \Psi_{k_i}| \leq \sum_{|\beta|=1} |k_i|^{-1} |\partial^\beta K_{j_i+\ell}^\alpha(t, \cdot) * \psi_{k_i}|, \quad (3.15)$$

where $K_{j_i+\ell}^\alpha(t, x)$ is the kernel of the operator $e^{-t(-\Delta)^\alpha} \Delta_{j_i+\ell}$. By the (2.4) in Lemma 2.1, we have the estimates

$$|\partial^\beta K_{j_i+\ell}^\alpha(t, x)| \lesssim e^{-ct|k_i|^{2\alpha}} 2^{|\beta|j_i} 2^{3j_i} (1 + 2^{j_i}|x|)^{-4}, \quad (3.16)$$

for some $c > 0$. Thus for any $\ell = 0, 1$ and $1 \leq i \leq \rho$, it follows from the (3.15) and (3.16) that

$$|\Delta_{j_i+\ell} e^{-t(-\Delta)^\alpha} \Psi_{k_i}| \lesssim e^{-ct|k_i|^{2\alpha}} M\psi \quad (3.17)$$

Hence in view of (3.14)–(3.17) and $|k_0| \leq |k_i|$, it immediately follows from Hardy-Littlewood maximal theorem that

$$\begin{aligned} \|u_1\|_{\dot{F}_{q_\alpha}^{-\alpha, r}} &\lesssim \frac{Q}{\sqrt{\rho}} e^{-ct|k_0|^{2\alpha}} \left\| \left(\sum_{i=1}^{\rho} \sum_{\ell=0,1} |(M\psi)(x)|^r \right)^{\frac{1}{r}} \right\|_{L_x^{q_\alpha}} \\ &\lesssim Q \rho^{\frac{1}{r} - \frac{1}{2}} e^{-ct|k_0|^{2\alpha}}. \end{aligned} \quad (3.18)$$

Thus we complete the proof for the cases $2 \leq r < \infty$. Finally, as $r = \infty$, similar to (3.11) and (3.14), we can obtain that

$$\left\| \sum_{s=1}^{\rho} |k_s|^\alpha \Psi_{k_s} \right\|_{\dot{F}_{q_\alpha}^{-\alpha, \infty}} \leq \left\| \sup_{1 \leq s \leq \rho} (|\Delta_{j_s} \Psi_{k_s}| + |\Delta_{j_s+1} \Psi_{k_s}|) \right\|_{L_x^{q_\alpha}}$$

and

$$\left\| \sum_{s=1}^{\rho} |k_s|^\alpha e^{-t(-\Delta)^\alpha} \Psi_{k_s} \right\|_{\dot{F}_{q_\alpha}^{-\alpha, \infty}} \leq \left\| \sup_{1 \leq s \leq \rho} \sum_{\ell=0,1} |\Delta_{j_s+\ell} e^{-t(-\Delta)^\alpha} \Psi_{k_s}| \right\|_{L_x^{q_\alpha}}$$

Hence by the estimates (3.12) and (3.17) we can immediately conclude the desired bounds of u_0 and u_1 for the case $r = \infty$. \square

Remark 3.3. From (3.10), for any given $\delta > 0$, if $r > 2$, then there exists sufficiently large ρ such that $\|u_0\|_{\dot{F}_{q_\alpha}^{-\alpha, r>2}} \leq \delta$ and $\|u_1\|_{\dot{F}_{q_\alpha}^{-\alpha, r>2}} \leq \delta$. Similarly, *it is also easy to prove that for any $q \in (q_\alpha, \infty]$,*

$$\|u_0\|_{\dot{F}_q^{-\alpha, r>2}} \lesssim Q \rho^{\frac{1}{r} - \frac{1}{2}}, \quad \|u_1\|_{\dot{F}_q^{-\alpha, r>2}} \lesssim Q \rho^{\frac{1}{r} - \frac{1}{2}}. \quad (3.19)$$

Therefore, for sufficiently large ρ , $\|u_0\|_{\dot{F}_q^{-\alpha, r>2}} \leq \delta$ and $\|u_1\|_{\dot{F}_q^{-\alpha, r>2}} \leq \delta$.

Lemma 3.4. For any $T > 0$, u_0 and u_1 given in (3.4) and (3.1), we obtain that

$$\|u_1\|_{L_x^{q_\alpha} L_T^2} \lesssim \frac{Q}{\sqrt{\rho}} (T^{\frac{1}{2}} |k_{\rho-N_0}|^\alpha + \sqrt{N_0}),$$

for any $0 \leq N_0 \leq \rho$. In particular, $\|u_1\|_{L_x^{q_\alpha} L_T^2} \rightarrow 0$ as $N_0 = 0$ and $T \rightarrow 0$.

Proof. From now on, we let $S_t u_0 = e^{-t(-\Delta)^\alpha} u_0 = u_1$. By the construction of initial data u_0 , it suffices to estimate $\sum_{s=1}^{\rho} |k_s|^\alpha S_t \Psi_{k_s}$. Similar to (3.11)), we have

$$\begin{aligned} \left\| \sum_{s=1}^{\rho} |k_s|^\alpha S_t \Psi_{k_s} \right\|_{L_x^{q_\alpha} L_T^2} &\lesssim \left\| \sum_{s=1}^{\rho} |k_s|^\alpha S_t \Psi_{k_s} \right\|_{L_T^2 \dot{F}_{q_\alpha}^{0,2}} \\ &\lesssim \left\| |M\psi| \left(\sum_{s=1}^{\rho} \sum_{\ell=0}^1 |k_s|^{2\alpha} e^{-ct|k_s|^{2\alpha}} \right)^{\frac{1}{2}} \right\|_{L_T^2 L_x^{q_\alpha}} \\ &\lesssim \left\| \left(\sum_{s=1}^{\rho} |k_s|^{2\alpha} e^{-ct|k_s|^{2\alpha}} \right)^{\frac{1}{2}} \right\|_{L_T^2} \left\| M\psi \right\|_{L_x^{q_\alpha}} \\ &\lesssim \left(\sum_{s=1}^{\rho} \int_0^T |k_s|^{2\alpha} e^{-ct|k_s|^{2\alpha}} dt \right)^{\frac{1}{2}}. \end{aligned} \quad (3.20)$$

Notice that

$$\int_0^T |k_s|^{2\alpha} e^{-ct|k_s|^{2\alpha}} dt \lesssim \min\{1, T|k_s|^{2\alpha}\} \text{ and } \sum_{s=1}^{\rho-N_0} |k_s|^{2\alpha} \lesssim |k_{\rho-N_0}|^{2\alpha},$$

then we get

$$(3.20) \lesssim \left(\sum_{s=1}^{\rho-N_0} T|k_s|^{2\alpha} + \sum_{s=\rho-N_0+1}^{\rho} 1 \right)^{\frac{1}{2}} \lesssim (T^{\frac{1}{2}} |k_{\rho-N_0}|^\alpha + \sqrt{N_0}). \quad (3.21)$$

Similar to (3.21), we can obtain the desired estimate. \square

By checking the estimates (3.21) for the case $N_0 = \rho$ again, we know that the best upper bound of $\|u_1\|_{L_x^{q_\alpha} L_T^2}$ is actually cQ , which is not good enough to bound the remainder $y(t, x)$ (below). Recalling the idea of estimating remainder term by means of bi-continuity of bilinear operator $B(u, v)$ in Lemma 2.2, it is also natural to hope that nonlinear terms of $y(t, x)$ are smaller. Therefore, we need to analyze how y evolve in different time scales and see their contributions by using the time-step-division method introduced by Bourgain-Pavlović in [1] to prove ill-posedness of the 3D incompressible Navier-Stokes equations. Let

$$|k_\rho|^{-2\alpha} = T_0 < T_1 < T_2 < \cdots < T_\beta = |k_0|^{-2\alpha}, \quad (3.22)$$

where $\beta = Q^3$, $T_\sigma = |k_{\rho_\sigma}|^{-2\alpha}$, $\rho_\sigma = \rho - \sigma Q^{-3} \rho$ and $\sigma = 0, 1, 2, \dots, \beta$.

Lemma 3.5. Assume that u_0 satisfy (3.4) and $u_1 = S_t u_0 = e^{-t(-\Delta)^\alpha} u_0$, we have

$$\left\| u_1 \right\|_{L_x^{q_\alpha} L_{[T_\sigma, T_{\sigma+1}]}^2} := \left\| u_1 \chi_{[T_\sigma, T_{\sigma+1}]}(t) \right\|_{L_x^{q_\alpha} L_{T_{\sigma+1}}^2} \lesssim \frac{Q}{\sqrt{\rho}} (1 + \sqrt{\rho Q^{-3}}). \quad (3.23)$$

Proof. Recall that $u_1 = S_t u_0$ and $u_0 = \frac{Q}{\sqrt{\rho}} \sum_{s=1}^{\rho} |k_s|^\alpha S_t (\Psi_{k_s} + \bar{\Psi}_{k_s} + \Phi_{k'_s} + \bar{\Phi}_{k'_s})$. It also suffices to estimate $\sum_{s=1}^{\rho} |k_s|^\alpha S_t \Psi_{k_s}$. Similar to (3.20), by using

$$\int_{T_\alpha}^{T_{\alpha+1}} |k_s|^{2\alpha} e^{-ct|k_s|^{2\alpha}} dt \lesssim \min \left\{ T_{\alpha+1} |k_s|^{2\alpha}, 1, e^{-cT_\alpha |k_s|^2} \right\},$$

and (3.22) we obtain that

$$\sum_{s=1}^{\rho_{\alpha+1}} T_{\alpha+1} |k_s|^2 \lesssim 1, \quad \sum_{s=\rho_{\alpha+1}+1}^{\rho_{\alpha}-1} 1 \lesssim \rho Q^{-3}, \quad \sum_{s=\rho_\alpha}^{\rho} e^{-T_\alpha |k_s|^2} \lesssim 1$$

and

$$\begin{aligned} \|u_0 \chi_{[T_\alpha, T_{\alpha+1}]}(t)\|_{L_x^{q_\alpha} L_{T_{\alpha+1}}^2} &\lesssim \frac{Q}{\sqrt{\rho}} \left(\sum_{s=1}^{\rho} \int_{T_\alpha}^{T_{\alpha+1}} e^{-ct|k_s|^{2\alpha}} |k_s|^{2\alpha} dt \right)^{\frac{1}{2}} \\ &\lesssim \frac{Q}{\sqrt{\rho}} (1 + \sqrt{\rho Q^{-3}}). \end{aligned} \quad (3.24)$$

Thus we prove (3.23). \square

The following result is a consequence of $\sum_{s=1}^{\rho} e^{-T_\beta |k_s|^2} \lesssim 1$ and Lemma 3.5.

Corollary 3.6. For any $T > T_\beta = |k_0|^{-2} = 2^{-2m_0}$, we have

$$\left\| u_1 \right\|_{L_x^{q_\alpha} L_{[T_\beta, T]}^2} = \left\| u_1 \chi_{[T_\beta, T]}(t) \right\|_{L_x^{q_\alpha} L_T^2} \lesssim \frac{Q}{\sqrt{\rho}}. \quad (3.25)$$

3.3 Estimates for the second approximation terms

We start this subsection by making some preliminary calculations. Recall that $u_1(\tau) = S_\tau u_0 = e^{\tau \Delta} u_0$. In order to study the bilinear form $u_2 = B(u_1, u_1)$, from the construction of initial data u_0 , we first split the *second approximation terms* u_2 into

$$u_2 = u_{2,0} + u_{2,1} + u_{2,2},$$

where

$$u_{2,0} = \frac{Q^2}{\rho} \sum_{s=1}^{\rho} \int_0^t |k_s|^{2\alpha} S_{t-\tau} \mathbb{P} F_s d\tau, \quad (3.26)$$

$$u_{2,1} = \frac{Q^2}{\rho} \sum_{s=1}^{\rho} \int_0^t |k_s|^{2\alpha} S_{t-\tau} \mathbb{P} G_s d\tau, \quad (3.27)$$

$$u_{2,2} = \frac{Q^2}{\rho} \sum_{s=1}^{\rho} \sum_{l \neq s} \int_0^t |k_s|^\alpha |k_l|^\alpha S_{t-\tau} \mathbb{P} H_{s,l} d\tau \quad (3.28)$$

and

$$\left\{ \begin{array}{l} F_s = S_\tau \Phi_{k'_s} \cdot \nabla S_\tau \Psi_{k_s} + S_\tau \bar{\Phi}_{k'_s} \cdot \nabla S_\tau \bar{\Psi}_{k_s} + S_\tau \Psi_{k_s} \cdot \nabla S_\tau \Phi_{k'_s} \\ \quad + S_\tau \bar{\Psi}_{k_s} \cdot \nabla S_\tau \bar{\Phi}_{k'_s} + S_\tau \Phi_{k'_s} \cdot \nabla S_\tau \bar{\Phi}_{k'_s} + S_\tau \bar{\Phi}_{k'_s} \cdot \nabla S_\tau \Phi_{k'_s} \\ \quad + S_\tau \Psi_{k_s} \cdot \nabla S_\tau \bar{\Psi}_{k_s} + S_\tau \bar{\Psi}_{k_s} \cdot \nabla S_\tau \Psi_{k_s}, \\ G_s = S_\tau \Phi_{k'_s} \cdot \nabla S_\tau \bar{\Psi}_{k_s} + S_\tau \bar{\Phi}_{k'_s} \cdot \nabla S_\tau \Psi_{k_s} + S_\tau \Psi_{k_s} \cdot \nabla S_\tau \bar{\Phi}_{k'_s} \\ \quad + S_\tau \bar{\Psi}_{k_s} \cdot \nabla S_\tau \Phi_{k'_s} + S_\tau \Phi_{k'_s} \cdot \nabla S_\tau \bar{\Phi}_{k'_s} + S_\tau \bar{\Phi}_{k'_s} \cdot \nabla S_\tau \bar{\Phi}_{k'_s} \\ \quad + S_\tau \Psi_{k_s} \cdot \nabla S_\tau \Psi_{k_s} + S_\tau \bar{\Psi}_{k_s} \cdot \nabla S_\tau \bar{\Psi}_{k_s}), \\ H_{s,l} = S_\tau (\Psi_{k_s} + \bar{\Psi}_{k_s} + \Phi_{k'_s} + \bar{\Phi}_{k'_s}) \cdot \nabla S_\tau (\Psi_{k_l} + \bar{\Psi}_{k_l} + \Phi_{k'_l} + \bar{\Phi}_{k'_l}). \end{array} \right. \quad (3.29)$$

Remark 3.7. Noticing that $u_{2,\ell}$ ($\ell = 0, 1, 2$) are *real-valued smooth* functions since they are summations of conjugated smooth functions. From (3.4) and (H1)–(H2), we observe that \widehat{F}_s , \widehat{G}_s and $\widehat{H}_{s,l}$ are *purely imaginary* since $\widehat{S_\tau \nabla \Phi_{k'_l}}$ is real-valued and $\widehat{S_\tau \Psi_{k_s}}$ is purely imaginary.

According to the different frequency interactions, we decompose the second approximation terms u_2 into three parts which are given in (3.26)–(3.28). Precisely,

- $u_{2,1}$ represents the *high-high to high* frequency interactions. This is always the best one, see Lemma 3.8.
- $u_{2,2}$ represents the *high-low to high* and *low-high to high* frequency interactions. Usually, these two kinds of frequency interactions can be well-controlled, see Lemma 3.9
- $u_{2,0}$ represents the *high-high to low* frequency interactions. This kind of interaction is always the worst one. In this paper, we will explore this part and gain the desired lower bound. Proof of the lower bound is a little complicated, thus we would like to give the details in the end of this subsection, see Lemma 3.11.

Now we prove the following estimates for $u_{2,1}$.

Lemma 3.8. *For any $1 < \alpha < \frac{5}{4}$, $r \geq 2$ and $q_\alpha = \frac{3}{\alpha-1}$, we have*

$$\|u_{2,1}\|_{L_T^\infty \dot{F}_{q_\alpha}^{-\alpha,r}} + \|u_{2,1}\|_{L_x^{q_\alpha} L_T^2} \lesssim \frac{Q^2}{\rho}. \quad (3.30)$$

Proof. First deal with the norm $\|u_{2,1}\|_{L_T^\infty \dot{F}_{d\alpha}^{-\alpha,r}}$. Recall that from the initial value construction in (3.4),

$$\text{supp } \widehat{\Psi}_{k_s} \subset B_{\frac{1}{4}}(k_s), \text{ supp } \widehat{\Phi}_{k'_s} \subset B_{\frac{1}{4}}(k'_s), \text{ supp } \widehat{\Psi}_{k_s} \subset B_{\frac{1}{4}}(-k_s), \text{ supp } \widehat{\Phi}_{k'_s} \subset B_{\frac{1}{4}}(-k'_s).$$

Hence for any $1 \leq s \leq \rho$, by (3.6)–(3.9), we have

$$G_s = (\Delta_{j_s+1} + \Delta_{j_s+2})G_s, \quad \Delta_j G_s = 0, \quad j \neq j_s + 1 \text{ and } j \neq j_s + 2. \quad (3.31)$$

which shows that $\{G_s\}_{s=1}^\rho$ is also a lacunary sequence. Therefore, by using boundedness of \mathbb{P} and $|k_s| \sim 2^{j_s}$, similar to (3.11), we obtain that

$$\begin{aligned} \|u_{2,1}\|_{L_T^\infty \dot{F}_{q\alpha}^{-\alpha,r}} &\lesssim \frac{Q^2}{\rho} \left\| \left\{ 2^{-\alpha j} \Delta_j \left(\sum_{s=1}^\rho \int_0^t |k_s|^{2\alpha} S_{t-\tau} G_s d\tau \right) \right\} \right\|_{l^r} \left\| \right\|_{L_T^\infty L_x^{q\alpha}} \\ &\lesssim \frac{Q^2}{\rho} \left\| \left(\sum_{s=1}^\rho \sum_{\ell=1,2} |k_s|^{\alpha r} \left| \int_0^t S_{t-\tau} \Delta_{j_s+\ell} G_s d\tau \right|^r \right)^{\frac{1}{r}} \right\|_{L_T^\infty L_x^{q\alpha}}. \end{aligned} \quad (3.32)$$

From Lemma 2.1 and similar to (3.17), we get

$$|(S_{t-\tau} \Delta_{j_s+\ell} G_s)(x)| \lesssim e^{-c(t-\tau)|k_s|^{2\alpha}} M_{|G_s|}(x). \quad (3.33)$$

where $M_{|G_s|}(x)$ denotes Hardy-Littlewood maximal function of $|G_s|$.

Note that $\Psi_{k_s} = (\Delta_{j_s} + \Delta_{j_s+1})\Psi_{k_s}$ and $\Psi_{k'_s} = (\Delta_{j_s} + \Delta_{j_s+1})\Psi_{k'_s}$. Hence by the (3.17), we have (if necessary, c can be adjusted to be smaller)

$$|S_\tau \Phi_{k'_s}| + |S_\tau \Psi_{k_s}| + |S_\tau \bar{\Phi}_{k'_s}| + |S_\tau \bar{\Psi}_{k_s}| \lesssim e^{-\frac{c}{2}\tau|k_s|^{2\alpha}} M\psi \quad (3.34)$$

and

$$|\nabla S_\tau \Phi_{k'_s}| + |\nabla S_\tau \Psi_{k_s}| + |\nabla S_\tau \bar{\Phi}_{k'_s}| + |\nabla S_\tau \bar{\Psi}_{k_s}| \lesssim |k_s| e^{-\frac{c}{2}\tau|k_s|^{2\alpha}} M\psi. \quad (3.35)$$

Thus

$$|G_s| \lesssim |S_\tau \Phi_{k'_s} \cdot \nabla S_\tau \bar{\Psi}_{k_s}| + \dots \lesssim |k_s| e^{-c\tau|k_s|^{2\alpha}} (M\psi)^2. \quad (3.36)$$

Let $\theta(x) = ((M\psi)(x))^2$. Then it follows from (3.33) and (3.36) that

$$|S_{t-\tau} \Delta_{j_s+\ell} G_s(x)| \lesssim |k_s| e^{-ct|k_s|^{2\alpha}} (M\theta)(x) \quad (3.37)$$

where $(M\theta)(x)$ is the maximal function of $\theta(x)$. Hence for any $1 \leq s \leq \rho$ and $\ell = 1, 2$, we have

$$\int_0^t |S_{t-\tau} \Delta_{j_s+\ell} G_s(x)| d\tau \lesssim t |k_s| e^{-ct|k_s|^{2\alpha}} (M\theta)(x) \lesssim |k_s|^{1-2\alpha} (M\theta)(x). \quad (3.38)$$

Now plugging (3.38) into (3.32) and recalling that $\alpha > 1$, then by Hardy-Littlewood maximal theorem we immediately obtain that

$$\|u_{2,1}\|_{L_T^\infty \dot{F}_{q_\alpha}^{-\alpha,2}} \lesssim \frac{Q^2}{\rho} \left\| \left(\sum_{s=1}^{\rho} |k_s|^{r(1-\alpha)} \right)^{\frac{1}{r}} M\theta \right\|_{L_x^{q_\alpha}} \lesssim \frac{Q^2}{\rho} \|\theta\|_{L_x^{q_\alpha}} \lesssim \frac{Q^2}{\rho}, \quad (3.39)$$

which concludes the desired bound of $\|u_{2,1}\|_{L_T^\infty \dot{F}_{q_\alpha}^{-\alpha,2}}$.

For the norm of $\|u_{2,1}\|_{L_x^{q_\alpha} L_T^2}$. Similar to (3.32), by using (3.37) we have

$$\begin{aligned} \|u_{2,1}\|_{L_x^{q_\alpha} L_T^2} &\leq \|u_{2,1}\|_{L_T^2 L_x^{q_\alpha}} = \|u_{2,1}\|_{L_T^2 \dot{F}_{q_\alpha}^{0,2}} \\ &\lesssim \frac{Q^2}{\rho} \left\| \left(\sum_{s=1}^{\rho} |k_s|^{2\alpha+1} t e^{-ct|k_s|^{2\alpha}} \right)^{\frac{1}{2}} (M\theta)(x) \right\|_{L_T^2 L_x^{q_\alpha}} \\ &\lesssim \frac{Q^2}{\rho} \left(\sum_{s=1}^{\rho} \int_0^\infty |k_s|^{2\alpha+1} t e^{-ct|k_s|^{2\alpha}} dt \right)^{\frac{1}{2}} \|M\theta\|_{L_x^{q_\alpha}} \\ &\lesssim \frac{Q^2}{\rho} \left(\sum_{s=1}^{\rho} k_s^{2(1-\alpha)} \right)^{\frac{1}{2}} \lesssim \frac{Q^2}{\rho}. \end{aligned} \quad (3.40)$$

Thus combining (3.39) and (3.40), we finish the proof of (3.30). \square

Next we estimate $u_{2,2}$.

Lemma 3.9. *For any $1 < \alpha < \frac{5}{4}$, $r \geq 2$ and $q_\alpha = \frac{3}{\alpha-1}$, we obtain that*

$$\|u_{2,2}\|_{L_T^\infty \dot{F}_{q_\alpha}^{-\alpha,r}} + \|u_{2,2}\|_{L_x^{q_\alpha} L_T^2} \lesssim \frac{Q^2}{\rho}. \quad (3.41)$$

Proof. In order to estimate $u_{2,2}$, we first rewrite it as follows:

$$\begin{aligned} u_{2,2} &= \frac{Q^2}{\rho} \sum_{s=1}^{\rho} |k_s|^\alpha \left(\int_0^t \left(\sum_{l < s} |k_l|^\alpha S_{t-\tau} \mathbb{P} H_{s,l} \right) d\tau \right. \\ &\quad \left. + \frac{Q^2}{\rho} \sum_{l=1}^{\rho} |k_l|^\alpha \int_0^t \left(\sum_{s < l} |k_s|^\alpha S_{t-\tau} \mathbb{P} H_{s,l} \right) d\tau \right). \end{aligned} \quad (3.42)$$

Recall from the initial value construction (3.4), we obtain that

$$\text{supp } \widehat{H}_{s,l} \subset B_{\frac{1}{2}}(\pm k_s \pm k_l) \cup B_{\frac{1}{2}}(\pm k_s \pm k'_l) \cup B_{\frac{1}{2}}(\pm k'_s \pm k_l) \cup B_{\frac{1}{2}}(\pm k'_s \pm k'_l).$$

As a result, we have

$$\begin{cases} \sum_{l < s} H_{s,l} = \sum_{l \leq s-1} (\Delta_{j_s} + \Delta_{j_{s+1}}) H_{s,l}, \quad \Delta_j H_{s,l} = 0, \quad j \neq j_s \text{ and } j \neq j_s + 1, \\ \sum_{s < l} H_{s,l} = \sum_{s \leq l-1} (\Delta_{j_l} + \Delta_{j_{l+1}}) H_{s,l}, \quad \Delta_j H_{s,l} = 0, \quad j \neq j_l \text{ and } j \neq j_l + 1. \end{cases} \quad (3.43)$$

By checking the estimates (3.34)–(3.35) and recalling that $\theta(x) = (M\psi)^2(x)$, for any $(s, l) \in (\mathbb{N} \cap [1, \rho]) \times (\mathbb{N} \cap [1, \rho])$, we get

$$|H_{s,l}| \lesssim |k_l| e^{-c\tau \max\{|k_s|^{2\alpha}, |k_l|^{2\alpha}\}} (M\theta)(x) \quad (3.44)$$

and for $\kappa = 0, 1$, similar to (3.33), by using (3.44) we get

$$|S_{t-\tau} \Delta_{\kappa+j_{\max\{s,l\}}} H_{s,l}|(x) \lesssim |k_l| e^{-ct \max\{|k_s|^{2\alpha}, |k_l|^{2\alpha}\}} (M\theta)(x). \quad (3.45)$$

In order to deduce the (3.41), it suffices to estimate the first part of the (3.42):

$$\left\| \frac{Q^2}{\rho} \sum_{s=1}^{\rho} \int_0^t |k_s|^\alpha \left(\sum_{l \leq s-1} |k_l|^\alpha S_{t-\tau} \mathbb{P} H_{s,l} \right) d\tau \right\|_{L_T^\infty \dot{F}_{q\alpha}^{-\alpha, r} \cap L_x^{q\alpha} L_T^2}.$$

Similar to (3.32), (3.39) and (3.45), by using boundedness of \mathbb{P} in $\dot{F}_{q\alpha}^{-\alpha, r}$, we have

$$\begin{aligned} & \left\| \frac{Q^2}{\rho} \sum_{s=1}^{\rho} \int_0^t |k_s|^\alpha \left(\sum_{l \leq s-1} |k_l|^\alpha S_{t-\tau} \mathbb{P} H_{s,l} \right) d\tau \right\|_{L_T^\infty \dot{F}_{q\alpha}^{-\alpha, r}} \\ & \lesssim \frac{Q^2}{\rho} \left\| \left(\sum_{s=1}^{\rho} \sum_{\kappa=0,1} \left(\sum_{l=1}^{s-1} |k_l|^\alpha \int_0^t S_{t-\tau} \Delta_{j_s+\kappa} H_{s,l} d\tau \right)^r \right)^{\frac{1}{r}} \right\|_{L_T^\infty L_x^{q\alpha}} \\ & \lesssim \frac{Q^2}{\rho} \left\| M\theta \left(\sum_{s=1}^{\rho} \left(\sum_{l=1}^{s-1} |k_l|^\alpha |k_s|^{1-2\alpha} \right)^r \right)^{\frac{1}{r}} \right\|_{L_x^{q\alpha}} \\ & \lesssim \frac{Q^2}{\rho} \left(\sum_{s=1}^{\rho} |k_{s-1}|^{\alpha r} |k_s|^{(1-2\alpha)r} \right)^{\frac{1}{r}} \lesssim \frac{Q^2}{\rho}, \end{aligned} \quad (3.46)$$

where we used

$$\sum_{l=1}^{s-1} |k_l|^\alpha \lesssim |k_{s-1}|^\alpha, \quad \left(\sum_{s=1}^{\rho} (|k_{s-1}|^{\alpha r} |k_s|^{(1-2\alpha)r})^{\frac{1}{r}} \right) \lesssim 1.$$

Following the similar arguments as in (3.40) and (3.46), we have

$$\left\| \frac{Q^2}{\rho} \sum_{s=1}^{\rho} \int_0^t |k_s|^\alpha \left(\sum_{l \leq s-1} |k_l|^\alpha S_{t-\tau} \mathbb{P} H_{s,l} \right) d\tau \right\|_{L_T^2 L_x^{q\alpha}} \lesssim \frac{Q^2}{\rho}.$$

Estimates for the second part of (3.42) follow in the similar way, hence we obtain the desired results. \square

Remark 3.10. By checking the proof of Lemmas 3.8 and 3.9, we obtain that the estimates for $u_{2,1}$ and $u_{2,2}$ are also true for any other $2 \leq q \leq \infty$.

Finally, we prove the lower bound of $u_{2,0}$ in critical space $\dot{F}_{q_\alpha}^{-\alpha, r > 2}$ and the upper bound of $u_{2,0}$ in its well-posed space. Specially, the lower bound obtained plays a *crucial* role in the proof of norm inflation.

To obtain such bounds, we will use several Fourier analysis methods. Due to the vector-valued nature of velocity field and the divergence free condition, we not only need to explore each of the three components but also need to analyze the action of Leray projection operator \mathbb{P} .

Lemma 3.11. *For $|k_0|^{-2\alpha} \ll T \ll 1$, $q_\alpha = \frac{3}{\alpha-1}$, $1 < \alpha < \frac{5}{4}$ and $2 \leq r \leq \infty$, we get*

$$\|u_{2,0}(T)\|_{\dot{F}_{q_\alpha}^{-\alpha, r}} \gtrsim Q^2, \quad (3.47)$$

$$\|u_{2,0}\|_{L_x^{q_\alpha} L_T^2} \lesssim T^{\frac{1}{2}} Q^2. \quad (3.48)$$

Proof. We first prove (3.47) and divide the proof into three steps.

Step 1. From (H1)–(H2), (3.26) and (3.29), we have

$$\text{supp } \widehat{u}_{2,0} \subset B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(k_s + k'_s) \cup B_{\frac{1}{2}}(-(k_s + k'_s)) \subset \left\{ |\xi| < \frac{8}{3} 2^2 \right\}, \quad (3.49)$$

where $\pm(k_s + k'_s) = (\pm 7, 0, 0)$. Hence for any $t > 0$, $u_{2,0}(x, t) \in \mathcal{C}^2(\mathbb{R}^3)$.

Step 2. From $\dot{F}_{q_\alpha}^{-\alpha, r} \hookrightarrow \dot{B}_{q_\alpha}^{-\alpha, \infty}$ and Definition 1.1, we have

$$\begin{aligned} \|u_{2,0}\|_{\dot{F}_{q_\alpha}^{-\alpha, r}} &\gtrsim \|u_{2,0}\|_{\dot{B}_{q_\alpha}^{-\alpha, \infty}} \gtrsim \|\Delta_2 u_{2,0}\|_{L_x^{q_\alpha}} + \|\Delta_3 u_{2,0}\|_{L_x^{q_\alpha}} \\ &\gtrsim \|(\Delta_2 + \Delta_3) u_{2,0}\|_{L_x^{q_\alpha}} \gtrsim \|(\Delta_2 + \Delta_3) u_{2,0}\|_{L_x^\infty}, \end{aligned} \quad (3.50)$$

where in the last inequality we used Bernstein's inequality.

Step 3. In this step, it suffices to prove

$$\left((\Delta_2 + \Delta_3) u_{2,0}^{[3]} \right) (-\pi/14, 0, 0, t) \gtrsim Q^2, \quad (3.51)$$

where $u_{2,0}^{[3]}$ denotes the third component of $u_{2,0}$. Once we prove (3.51), then combining $u_{2,0} \in \mathcal{C}^2(\mathbb{R}^3)$ with (3.50), we obtain that

$$\|u_{2,0}\|_{\dot{F}_{q_\alpha}^{-\alpha, r}} \gtrsim \|(\Delta_2 + \Delta_3) u_{2,0}^{[3]}\|_{L_x^\infty} \gtrsim Q^2,$$

which is the desired (3.47).

To prove (3.51), we recall that from Remark 3.7, \widehat{F}_s is purely imaginary, $\widehat{\mathbb{P}}$ and $u_{2,0}$ are real-valued, $(\Delta_2 + \Delta_3) u_{2,0}^{[3]}(x, t)$ can be rewritten as

$$(\Delta_2 + \Delta_3) u_{2,0}^{[3]}(x, t) = \frac{Q^2}{(2\pi)^3 \rho} \sum_{s=1}^{\rho} \int_0^t \int_{\mathbb{R}^3} \frac{i \sin(x \cdot \xi) |k_s|^{2\alpha}}{e^{(t-\tau)|\xi|^{2\alpha}}} \widehat{\mathbb{P}}_3(\xi) \cdot ((\Delta_2 + \Delta_3) F_s)^\wedge(\xi) d\tau d\xi$$

$$:= \frac{Q^2}{(2\pi)^3 \rho} \sum_{s=1}^{\rho} I_s(x, t) \quad (3.52)$$

where $\widehat{\mathbb{P}}_3(\xi) = (\frac{-\xi_3 \xi_1}{|\xi|^2}, \frac{-\xi_3 \xi_2}{|\xi|^2}, \frac{|\xi_1|^2 + |\xi_2|^2}{|\xi|^2})$ is the third row vector of $\widehat{\mathbb{P}} = I_d - \frac{\xi \otimes \xi}{|\xi|^2}$.

Later on, we will show that given $x_0 = (-\frac{\pi}{14}, 0, 0)$, there exists positive constant δ such that for any $1 \leq s \leq \rho$ and $\frac{1}{|k_0|^{2\alpha}} \ll t \ll 1$,

$$I_s(x_0, t) \geq \delta.$$

Noticing that $B_{\frac{1}{2}}(0) \cap \{\frac{3}{4} 2^2 < |\xi| < \frac{8}{3} 2^3\} = \emptyset$ and

$$(\Delta_2 + \Delta_3)F_s = S_\tau \Phi_{k'_s} \cdot \nabla S_\tau \Psi_{k_s} + S_\tau \bar{\Phi}_{k'_s} \cdot \nabla S_\tau \bar{\Psi}_{k_s} + S_\tau \Psi_{k_s} \cdot \nabla S_\tau \Phi_{k'_s} + S_\tau \bar{\Psi}_{k_s} \cdot \nabla S_\tau \bar{\Phi}_{k'_s},$$

then we get

$$I_s(x, t) = \sum_{\ell=1}^4 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} A_s(x, t, \xi, \eta) B_{\ell, s}(\xi, \eta) d\eta d\xi := \sum_{\ell=1}^4 I_{s, \ell}(x, t), \quad (3.53)$$

where

$$\begin{aligned} A_s(x, t, \xi, \eta) &:= \sin(-x \cdot \xi) \int_0^t |k_s|^{2\alpha} e^{-t|\xi|^{2\alpha} + \tau(|\xi|^{2\alpha} - |\xi - \eta|^{2\alpha} - |\eta|^{2\alpha})} d\tau \\ &= \frac{|k_s|^{2\alpha} \sin(-x \cdot \xi) (e^{-t|\xi|^{2\alpha}} - e^{-t(|\xi - \eta|^{2\alpha} + |\eta|^{2\alpha})})}{|\xi - \eta|^{2\alpha} + |\eta|^{2\alpha} - |\xi|^{2\alpha}} \end{aligned} \quad (3.54)$$

and

$$\left\{ \begin{aligned} B_{1, s}(\xi, \eta) &= \widehat{\Phi}_{k'_s}(\xi - \eta) \cdot \eta \widehat{\mathbb{P}}_3(\xi) \cdot \widehat{\Psi}_{k_s}(\eta), \\ &= \widehat{\psi}_{k'_s}(\xi - \eta) \widehat{\psi}_{k_s}(\eta) \frac{(\xi_1 \eta_2 - \xi_2 \eta_1) (\xi_1^2 \eta_2 + \xi_2^2 \eta_2 + \xi_3 \xi_2 \eta_3)}{|\xi|^2 |k_s|^2}, \\ B_{3, s}(\xi, \eta) &= \widehat{\Psi}_{k_s}(\xi - \eta) \cdot \eta \widehat{\mathbb{P}}_3(\xi) \cdot \widehat{\Phi}_{k'_s}(\eta), \\ &= \widehat{\psi}_{k_s}(\xi - \eta) \widehat{\psi}_{k'_s}(\eta) \frac{(\xi_3 \eta_2 - \xi_2 \eta_3) (\xi_3 \xi_2 \eta_1 - \xi_3 \xi_1 \eta_2)}{|\xi|^2 |k_s|^2}, \\ B_{2, s}(\xi, \eta) &= -B_{1, s}(-\xi, -\eta), \quad B_{4, s}(\xi, \eta) = -B_{3, s}(-\xi, -\eta). \end{aligned} \right. \quad (3.55)$$

From (3.53)–(3.55), we observe that

$$I_{s, 1}(x, t) = I_{s, 2}(x, t), \quad I_{s, 3}(x, t) = I_{s, 4}(x, t). \quad (3.56)$$

Denoting $\tilde{\eta} := \eta - k_s \in B_{1/4}(0)$, $\xi_0 := k_s + k'_s = (7, 0, 0)$ and recalling that $x_0 = (-\frac{\pi}{14}, 0, 0)$, we get

$$I_{s, 1}(x_0, t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \widehat{\psi}(\xi - \xi_0 - \tilde{\eta}) \widehat{\psi}(\tilde{\eta}) C_{s, 1}(\xi, \tilde{\eta}) A_s(x_0, t, \xi, \tilde{\eta} + k_s) d\xi d\tilde{\eta},$$

where

$$C_{s,1}(\xi, \tilde{\eta}) = \frac{(\xi_1(\tilde{\eta}_2 + |k_s|) - \xi_2\tilde{\eta}_1)((\xi_1^2 + \xi_2^2)(\tilde{\eta}_2 + |k_s|) + \xi_3\xi_2\tilde{\eta}_3)}{|\xi|^2|k_s|^2}. \quad (3.57)$$

Since $|\tilde{\eta}| < \frac{1}{4}$, $|\xi| < |\xi_0| + \frac{1}{2}$ and $2^{m_0} = |k_0| < |k_s| = 2^{\frac{(s+1)(s+2m_0)}{2}}$, as m_0 tends to infinity, we have

$$\lim_{m_0 \rightarrow \infty} C_{s,1}(\xi, \tilde{\eta}) = \frac{\xi_1(\xi_1^2 + \xi_2^2)}{|\xi|^2}$$

and for $\frac{1}{|k_0|^{2\alpha}} \ll t \ll 1$,

$$\lim_{m_0 \rightarrow \infty} A_s(x_0, t, \xi, \tilde{\eta} + k_s) = \frac{e^{-t|\xi|^{2\alpha}} \sin(\frac{\pi\xi_1}{14})}{2}.$$

As a consequence,

$$\lim_{m_0 \rightarrow \infty} (I_{s,1} + I_{s,2}) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \widehat{\psi}(\xi - \xi_0 - \tilde{\eta}) \widehat{\psi}(\tilde{\eta}) \frac{\xi_1(\xi_1^2 + \xi_2^2) \sin(\frac{\pi\xi_1}{14})}{|\xi|^2} e^{-t|\xi|^{2\alpha}} d\xi d\tilde{\eta}.$$

Similarly, we have

$$\lim_{m_0 \rightarrow \infty} (I_{s,3} + I_{s,4}) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \widehat{\psi}(\xi - \xi_0 - \tilde{\eta}) \widehat{\psi}(\tilde{\eta}) \frac{-\xi_1\xi_3^2 \sin(\frac{\pi\xi_1}{14})}{|\xi|^2} e^{-t|\xi|^{2\alpha}} d\xi d\tilde{\eta}.$$

For any $\xi \in B_{1/2}(\xi_0) \cup B_{1/2}(-\xi_0)$, $\max\{|\xi_1 \pm 7|, |\xi_2|, |\xi_3|\} < 1/2$, there exists absolute positive constants δ and N_0 such that if $m_0 > N_0$, then for any $1 \leq s \leq \rho$,

$$I_s(x_0, t) = \sum_{\ell=1}^4 I_{s,\ell}(x_0, t) \geq \delta \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \widehat{\psi}(\xi - \xi_0 - \tilde{\eta}) \widehat{\psi}(\tilde{\eta}) d\xi d\tilde{\eta} = \delta,$$

which concludes the key estimate (3.51) by using (3.52).

It remains to prove (3.48), i.e. $\|u_{2,0}\|_{L_x^{q\alpha} L_T^2} \lesssim \|u_{2,0}\|_{L_T^2 L_x^{q\alpha}} \lesssim Q^2 T^{\frac{1}{2}}$. We need to estimate $u_{2,0}^{[1]}$, $u_{2,0}^{[2]}$ and $u_{2,0}^{[3]}$. Similar to (3.52), we have

$$\widehat{u}_{2,0}^{[3]}(\xi, t) = \frac{Q^2}{(2\pi)^{\frac{3}{2}}\rho} \sum_{s=1}^{\rho} \int_0^t |k_s|^{2\alpha} e^{-(t-\tau)|\xi|^{2\alpha}} \widehat{\mathbb{P}}_3(\xi) \cdot \widehat{F}_s(\xi) d\tau. \quad (3.58)$$

By using divergence free condition of Ψ_{k_s} , $\Phi_{k'_s}$ and $|\xi| < \frac{8}{3} 2^2$ as well as $\widehat{\psi}(\cdot) \geq 0$, it is easy to show that, for instance $S_\tau \Psi_{k_s} \cdot \nabla S_\tau \Phi_{k'_s} = \nabla \cdot (S_\tau \Psi_{k_s} \otimes S_\tau \Phi_{k'_s})$. Thus we get

$$e^{-(t-\tau)|\xi|^{2\alpha}} |\widehat{\mathbb{P}}_3(\xi) \cdot \widehat{F}_s(\xi)| \lesssim e^{-c\tau|k_s|^{2\alpha}} f(\xi; k_s, k'_s), \quad (3.59)$$

where $f(\xi; k_s, k'_s) := (\widehat{\psi}_{k_s} * \widehat{\psi}_{k'_s} + \widehat{\psi}_{k_s} * \widehat{\psi}_{k_s} + \widehat{\psi}_{k'_s} * \widehat{\psi}_{k'_s})(\xi)$.

Combining (3.58)–(3.59), we observe that for any $t > 0$,

$$|\widehat{u}_{2,0}^{[3]}(\xi, t)| \lesssim \frac{Q^2}{\rho} \sum_{s=1}^{\rho} \int_0^t |k_s|^{2\alpha} e^{-c\tau|k_s|^{2\alpha}} d\tau f(\xi; k_s, k'_s) \lesssim Q^2 f(\xi; k_s, k'_s).$$

Similarly, we get $|\widehat{u}_{2,0}^{[1]}| + |\widehat{u}_{2,0}^{[2]}| \lesssim Q^2 f(\xi; k_s, k'_s)$. By applying Hausdorff-Young's inequality and $\|\widehat{\psi}\|_{L_\xi^1 \cap L_\xi^{\frac{q\alpha}{q\alpha-1}}} \lesssim 1$ in (H2) to $f(\xi; k_s, k'_s)$, we have

$$\|u_{2,0}\|_{L_T^2 L_x^{q\alpha}} \lesssim \|\widehat{u}_{2,0}\|_{L_T^2 L_\xi^{\frac{q\alpha}{q\alpha-1}}} \lesssim Q^2 \left\| \|f(\xi; k_s, k'_s)\|_{L_\xi^{\frac{q\alpha}{q\alpha-1}}} \right\|_{L_T^2} \lesssim Q^2 T^{\frac{1}{2}}.$$

Therefore, we complete the proof. \square

3.4 Estimates of remainder y

In this subsection, we use iteration arguments to prove the *a-priori* estimate for remainder y . Recall that y satisfy the integral equations (3.2), i.e.

$$y = G_0 + G_1 - G_2$$

with initial condition $y|_{t=0} = 0$ and

$$G_2 = B(y, y), G_1 = B(y, u_2 - u_1) + B(u_2 - u_1, y), G_0 = B(u_2, u_1 - u_2) + B(u_1, u_2).$$

From Lemma 3.5, we observe that in order to obtain more accurate decay estimate for y , it suffices to split u_1 , u_2 and y into two terms, e.g.

$$\begin{cases} u_1 = u_1 \chi_{[0, T_\sigma]}(t) + u_1 \chi_{[T_\sigma, T_{\sigma+1}]}(t), \\ u_2 = u_2 \chi_{[0, T_\sigma]}(t) + u_2 \chi_{[T_\sigma, T_{\sigma+1}]}(t), \\ y = y \chi_{[0, T_\sigma]}(t) + y \chi_{[T_\sigma, T_{\sigma+1}]}(t), \end{cases}$$

Plugging the above decompositions of u_1 , u_2 and y into G_0 , G_1 and G_2 , we have the following iteration rules which play an important role in controlling y .

Lemma 3.12. *If y solves system (3.2)–(3.3), then for any $\sigma = 0, 1, 2, \dots, Q^3$ and for large enough ρ and $|k_0|$ we have*

$$\|y\|_{X_{T_{\sigma+1}}} \lesssim Q^{\sigma+3} (\rho^{-1} + |k_0|^{-\alpha}). \quad (3.60)$$

Moreover, for any $T > |k_0|^{-2\alpha}$, we have

$$\|y\|_{X_T} \lesssim Q^3 (\rho^{-1} + T^{\frac{1}{2}}) + Q^{Q^3+3} (\rho^{-1} + |k_0|^{-\alpha}). \quad (3.61)$$

Proof. Applying Lemma 2.2 to (3.2)–(3.3), we have the following bilinear estimates:

$$\begin{aligned} \|y\|_{X_{T_{\sigma+1}}} &\lesssim \|u_2\|_{X_{T_{\sigma+1}}} (\|u_1\|_{X_{T_{\sigma+1}}} + \|u_2\|_{X_{T_{\sigma+1}}}) + (\|u_1\|_{X_{T_{\sigma+1}}} + \|u_2\|_{X_{T_{\sigma+1}}}) \|y\|_{X_{T_{\sigma}}} \\ &\quad + (\|u_1\|_{X_{[T_{\sigma}, T_{\sigma+1}]}} + \|u_2\|_{X_{[T_{\sigma}, T_{\sigma+1}]}}) \|y\|_{X_{T_{\sigma+1}}} + \|y\|_{X_{T_{\sigma+1}}}^2. \end{aligned} \quad (3.62)$$

Recalling that for any $1 \leq \sigma \leq \beta$, $T_{\sigma} \leq T_{\beta}$. Then from Lemmas 3.4–3.11, we get

$$\|u_2\|_{X_{T_{\sigma+1}}} \lesssim Q^2 \rho^{-1} + Q^2 T_{\beta}^{\frac{1}{2}}, \quad \|u_1\|_{X_{T_{\sigma}}} \lesssim Q, \quad \|u_1\|_{X_{[T_{\sigma}, T_{\sigma+1}]}} \lesssim Q^{-\frac{1}{2}}. \quad (3.63)$$

Plugging (3.63) in (3.62), and assuming that $\rho > Q^5$, $T_{\beta} = |k_0|^{-2\alpha} < Q^{-5}$, we have

$$\begin{aligned} \|y\|_{X_{T_{\sigma+1}}} &\lesssim (Q^2 \rho^{-1} + Q^2 T_{\beta}^{\frac{1}{2}})(Q + Q^2 \rho^{-1} + Q^2 T_{\beta}^{\frac{1}{2}}) + (Q + Q^2 \rho^{-1} + Q^2 T_{\beta}^{\frac{1}{2}}) \|y\|_{X_{T_{\sigma}}} \\ &\quad + (Q^{-\frac{1}{2}} + Q^2 \rho^{-1} + Q^2 T_{\beta}^{\frac{1}{2}}) \|y\|_{X_{T_{\sigma+1}}} + \|y\|_{X_{T_{\sigma+1}}}^2 \\ &\lesssim Q^3 (\rho^{-1} + |k_0|^{-\alpha}) + Q \|y\|_{X_{T_{\sigma}}} + Q^{-\frac{1}{2}} \|y\|_{X_{T_{\sigma+1}}} + \|y\|_{X_{T_{\sigma+1}}}^2. \end{aligned} \quad (3.64)$$

Similarly, when $T > T_{\beta}$, by splitting $[0, T]$ into $[0, T_{\beta}]$ and $[T_{\beta}, T]$, then using Corollary 3.6 and (3.62)–(3.64), we get $\|u_1\|_{X_{[T_{\beta}, T]}} \lesssim Q \rho^{-\frac{1}{2}}$ and

$$\|y\|_{X_T} \lesssim Q^3 (\rho^{-1} + T^{\frac{1}{2}}) + Q \|y\|_{X_{T_{\beta}}} + Q^{-\frac{1}{2}} \|y\|_{X_T} + \|y\|_{X_T}^2. \quad (3.65)$$

Lemma 3.4 ensures that $\|y\|_{X_{T_0}}$ can be small enough since $T_0 = |k_{\rho}|^{-2\alpha}$ and

$$\|u_1\|_{X_{T_0}} \lesssim Q \rho^{-\frac{1}{2}} T_0^{\frac{1}{2}} |k_{\rho}|^{\alpha} \lesssim Q \rho^{-\frac{1}{2}}$$

and ρ is large enough. Thus iteration argument can be applied to (3.64)–(3.65) and hence we obtain the desired results. \square

Making use of Lemmas 2.2 and 3.12, we obtain the following estimate.

Corollary 3.13. *For any $r > 2$, sufficiently large ρ and $|k_0|$ such that $\rho \gg Q^{Q^3+2}$, $|k_0|^{-\alpha} \ll Q^{-Q^3-2}$ and $|k_0|^{-2\alpha} < T \ll Q^{-4}$, we have*

$$\|y(T)\|_{\dot{F}_{q_{\alpha}}^{-\alpha, r}} \ll Q^2. \quad (3.66)$$

Proof. From (3.2)–(3.3), we notice that $y(T) = G_0(T) + G_1(T) - G_2(T)$ and $G_i(T)$ are several bilinear terms. By applying Lemma 2.2, we obtain that

$$\begin{aligned} \|y(T)\|_{\dot{F}_{q_{\alpha}}^{-\alpha, r}} &\lesssim \|y(T)\|_{\dot{F}_{q_{\alpha}}^{-\alpha, 2}} \lesssim \|y\|_{L_T^{\infty} \dot{F}_{q_{\alpha}}^{-\alpha, 2}} \\ &\lesssim \|u_2\|_{X_T} (\|u_1\|_{X_T} + \|u_2\|_{X_T}) + (\|u_1\|_{X_T} + \|u_2\|_{X_T}) \|y\|_{X_T} + \|y\|_{X_T}^2. \end{aligned}$$

Applying Lemmas 3.4, 3.8–3.12 to the above inequality, we have

$$\|y(T)\|_{\dot{F}_{q_{\alpha}}^{-\alpha, r}} \lesssim (Q^2 \rho^{-1} + Q^2 T^{\frac{1}{2}})(Q + Q^2 \rho^{-1} + Q^2 T^{\frac{1}{2}})$$

$$\begin{aligned}
& + (Q + Q^2 \rho^{-1} + Q^2 T^{\frac{1}{2}}) \left(Q^3 (\rho^{-1} + T^{\frac{1}{2}}) + Q^{Q^3+3} (\rho^{-1} + |k_0|^{-\alpha}) \right) \\
& + \left(Q^3 (\rho^{-1} + T^{\frac{1}{2}}) + Q^{Q^3+3} (\rho^{-1} + |k_0|^{-\alpha}) \right)^2 \\
& \ll Q^2.
\end{aligned}$$

Hence we prove the desired result. \square

3.5 Proof of Theorem 1.3

In this subsection, combining the results proved in Subsections 3.1–3.4, we are ready to prove the ill-posedness of the gNS by showing norm inflation.

Proof of Theorem 1.3. Combining the equalities (3.1) and (3.26)–(3.28), the estimates (3.10), (3.30), (3.41), (3.47) and (3.66), we have

$$\begin{aligned}
\|u(T)\|_{\dot{F}_{q\alpha}^{-\alpha,r}} & \geq \|u_{2,0}(T)\|_{\dot{F}_{q\alpha}^{-\alpha,r}} \\
& - \left(\|u_1(T)\|_{\dot{F}_{q\alpha}^{-\alpha,r}} + \|u_{2,1}(T)\|_{\dot{F}_{q\alpha}^{-\alpha,r}} + \|u_{2,2}(T)\|_{\dot{F}_{q\alpha}^{-\alpha,r}} + \|y(T)\|_{\dot{F}_{q\alpha}^{-\alpha,r}} \right) \\
& \gtrsim \|(\Delta_2 + \Delta_3)u_{2,0}^{[3]}(T)\|_{L_x^\infty} \\
& - \left(\|u_1(T)\|_{\dot{F}_{q\alpha}^{-\alpha,r}} + \|u_{2,1}(T)\|_{\dot{F}_{q\alpha}^{-\alpha,2}} + \|u_{2,2}(T)\|_{\dot{F}_{q\alpha}^{-\alpha,2}} + \|y(T)\|_{\dot{F}_{q\alpha}^{-\alpha,2}} \right) \\
& \gtrsim Q^2 \left(1 - Q^{-1} \rho^{\frac{1}{r}-\frac{1}{2}} - \rho^{-1} - o(1) \right) \gtrsim Q^2,
\end{aligned}$$

where $0 < o(1) \ll \frac{1}{2}$, $\rho \gg Q^{Q^3+2}$ and $|k_0|^{-2\alpha} < T \ll Q^{-4}$. Hence we finish the proof.

Appendix

In this appendix, we will give a proof of Lemma 2.4 and state some extensions for reader's convenience. In fact, the equivalent estimates (2.10) of Lemma 2.4 can be immediately concluded from the following general Littlewood-Paley g-function characterizations of $L^p(\mathbb{R}^n)$, which in turn base on the vector-value singular integrals theory, see e.g. Stein [19, p.46 and p.185].

Lemma A.1. *Let $\Phi(x)$ be any function on \mathbb{R}^n satisfying $\int_{\mathbb{R}^n} \Phi(x) dx = 0$ and*

$$|\Phi(x)| + |\nabla \Phi(x)| \leq A(1 + |x|)^{-n-1}, \quad (\text{A.1})$$

for some constant $A > 0$. Then for any $1 < q < \infty$ the following estimate

$$\|s_\Phi f\|_{L_x^q} := \left\| \left(\int_0^\infty |\Phi_t * f|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L_x^q} \leq C_q \|f\|_{L^q} \quad (\text{A.2})$$

holds for $\Phi_t(x) = \frac{1}{t^n} \Phi(\frac{x}{t})$ with $t > 0$. Furthermore, if Φ is nondegenerate, in the sense that there exists a function Ψ satisfying the same conditions as Φ such that

$$\int_0^\infty \widehat{\Phi}(t\xi) \widehat{\Psi}(t\xi) \frac{dt}{t} = 1, \quad \xi \neq 0, \quad (\text{A.3})$$

then the converse inequality $\|f\|_{L_x^q} \leq C'_q \|s_\Phi f\|_{L_x^q}$ holds for any $1 < q < \infty$.

Proof of Lemma A.1 Let $\Phi(x) = \mathcal{F}^{-1}(|\cdot|^\alpha e^{-|\cdot|^{2\alpha}})(x)$ for any $\alpha \geq 1$. Then

$$\int_{\mathbb{R}^n} \Phi(x) dx = \widehat{\Phi}(0) = 0$$

and it is easy to check that $\Phi(x)$ satisfies the condition (A.1) by the similar argument as done in Lemma 2.2. Moreover, we can choose $\Psi = c\Phi$ for some $c > 0$ such that

$$\int_0^\infty |\widehat{\Phi}(t\xi)|^2 \frac{dt}{t} = 1/c, \quad \xi \neq 0,$$

which means the Φ is non-degenerate. Thus it follows from Theorem 3.5 that

$$\left\| \left(\int_0^\infty |t^\alpha (-\Delta)^{\alpha/2} e^{-t^{2\alpha} (-\Delta)^\alpha} h|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L_x^q} \sim \|h\|_{L_x^q} \quad (\text{A.4})$$

Equivalently, by the variable t changing and set $h = (-\Delta)^{-\alpha/2} f$, we have

$$\left\| \left(\int_0^\infty |e^{-t(-\Delta)^\alpha} f|^2 dt \right)^{1/2} \right\|_{L_x^q} \sim \|(-\Delta)^{-\alpha/2} f\|_{L_x^q}.$$

Note that by Littlewood-Plaley theorem and isomorphism, it follows that

$$\|(-\Delta)^{-\alpha/2} f\|_{L_x^q} \sim \|(-\Delta)^{-\alpha/2} f\|_{\dot{F}_q^{0,2}} \sim \|f\|_{\dot{F}_q^{-\alpha,2}}$$

for any $1 < q < \infty$, hence, the fractional semigroup characterization

$$\left\| \left(\int_0^\infty |e^{-t(-\Delta)^\alpha} f|^2 dt \right)^{1/2} \right\|_{L_x^q} \sim \|f\|_{\dot{F}_q^{-\alpha,2}} \quad (\text{A.5})$$

holds for any $1 < q < \infty$. When $n = 3$, $1 < \alpha < \frac{5}{4}$ and $q_\alpha = \frac{3}{\alpha-1}$, we can immediately get the desired results in Lemma 2.4.

Remark 3.14. Intrinsically, we can extend also the estimate (A.5) to general case, for instance, for any $s < 0$ and $1 < r, q < \infty$,

$$\left\| \left(\int_0^\infty |t^{-\frac{s}{2\alpha}} e^{-t(-\Delta)^\alpha} f|^r \frac{dt}{t} \right)^{1/r} \right\|_{L_x^q} \sim \|f\|_{\dot{F}_q^{s,r}}. \quad (\text{A.6})$$

holds. In particular, when $s = -\alpha$ and $r = 2$, we immediately obtain the estimate (A.5) above. However, it should be pointed out that the proof of general estimate (A.6) is different and more involved than the special index $r = 2$, essentially depending on a vector-valued version of maximal functions inequality, originally due to Fefferman and Stein. In this deep connection, one can see Triebel book [20, p. 101] for many general characterizations of nonhomogeneous Triebel-Lizorkin space $F_q^{s,r}(\mathbb{R}^n)$, where one can check similar methods also work well for the proof of the homogeneous type (A.6). Hence we omit these details in the appendix for concision.

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